THE

MATHEMATICAL GAZETTE

EDITED BY

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OUGHTRED'S "CLAVIS".

OUR Association has numbered among its members several authors whose textbooks have won world-wide repute. To include in our series of plates some references to earlier influential textbooks is therefore not inappropriate.

William Oughtred (1574–1660) is best known for his *Clavis Mathematicae* (1631; second edition, 1648; third edition, much enlarged, 1652). Soon after publication it had been recognised as a standard work on arithmetic and algebra, containing practically all that was then known on these subjects; it was used at Oxford and at Cambridge as a text on which lectures were given. Newton records that he read "clean over Oughtred's *Clavis*" somewhere about 1663, and Robert Boyle speaks highly of it. Its influence lasted through the century, and has been felt much further, for it can be regarded as setting a standard of exposition which entitles it to be reckoned the first modern textbook in this country.

Oughtred paid particular attention to the use of symbols, and was the first to employ the St. Andrew's cross as a sign for multiplication. He also was a pioneer in the use of abbreviations for the sine, tangent, cosine and cotangent, probably the inventor of the radix method of calculating logarithms, and, according to Cajori, the inventor of the slide rule.

Oughtred was born at Eton, and educated at Eton and King's College, Cambridge; he took orders in 1603 and spent the rest of his life in country livings. Several anecdotes of him survive. His wife was, we are told, a penurious shrew, who would often deprive him of his after-supper candle, "whereby many a good notion was lost and many a problem unsolved". Oughtred died a few weeks after the restoration of Charles II, and his death is said to have been caused by his immoderate joy at this event; but the weight of his eighty-six years may have been as much the cause as his exuberant Royalism.

For the plate, we are indebted to the Photographic Department of the British Museum.

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AN EXTENDED USE OF VELOCITY-TIME GRAPHS.

By J. N. Banks.

Most of the standard textbooks and many teachers of Applied Mathematics make use of a velocity-time graph to facilitate the solution of elementary kinematical problems including vertical motion under gravity, but it is unusual to extend the method to projectiles and allied problems, in spite of the ease with which the more usual type of problem can be solved. Such problems may be solved by simple calculations based upon the properties of similar triangles and the areas of rectangles, triangles and trapezia, that is, by the normal velocity-time graph procedure.

Many advanced problems can be solved easily by such methods, including questions on projectiles, impact and the impulsive motion of connected particles. Those on projectiles may be solved by drawing velocity-time graphs for the motion perpendicular and parallel to the inclined plane in question, using an equal time base in each case. In those concerning impulses, the sudden change in velocity can be indicated clearly, including the reversal due to direct impact on a plane surface. The motion of two connected particles is illustrated by drawing a graph for each particle.

The examples presented are of the more difficult type, such as appear in Higher School Certificate and Scholarship papers. In most cases the application to a more general problem is indicated, but the method is particularly useful when applied to numerical examples.

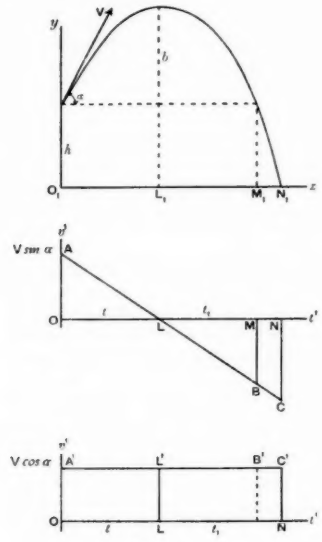


Fig. 1.

1. A gun is fired from the top of a cliff of height h and the shot attains a maximum height of $h + b$ above sea level, and strikes the sea at a distance a

from the foot of the cliff. Prove that the angle of elevation of the gun is given by α where

$$a^2 \tan^2 \alpha - 4ab \tan \alpha - 4bh = 0.$$

The (v, t) graph for vertical motion is $ALBC$, hence from the triangle AOL we have

$$V \sin \alpha = gt. \quad \text{.....(1)}$$

$$\text{Area } \triangle AOL = b = \tfrac{1}{2} V \sin \alpha \,.\, t = \tfrac{1}{2} gt^2. \quad \text{.....(2)}$$

From the triangle LCN, $\quad h + b = \tfrac{1}{2} g t_1^2. \quad \text{.....(3)}$

The (v, t) graph for horizontal motion is given by $A'L'B'C'$.
Hence we have

$$\text{Area } A'ONC' = a = V \cos \alpha . (t + t_1). \quad \text{.....(4)}$$

Multiply four times equation (2) by equation (3) :

$$4b(h + b) = g^2 t^2 t_1^2. \quad \text{.....(5)}$$

Divide equation (1) by equation (4) :

$$a \tan \alpha = gt(t + t_1).$$

Thus $\quad a \tan \alpha - 2b = gt(t + t_1) - gt^2 = gtt_1,$

and so $\quad (a \tan \alpha - 2b)^2 = g^2 t^2 t_1^2 = 4b(h + b),$

whence $\quad a^2 \tan^2 \alpha - 4ab \tan \alpha - 4bh = 0.$

2. A particle is projected with velocity u so as to strike at right angles a plane through the point of projection inclined at angle β to the horizon. Show that the range on this plane is $\dfrac{2u^2 \sin \beta}{g(1 + 3 \sin^2 \beta)}$, and that the angle of projection is given by $2 \tan \alpha = \cot \beta$.

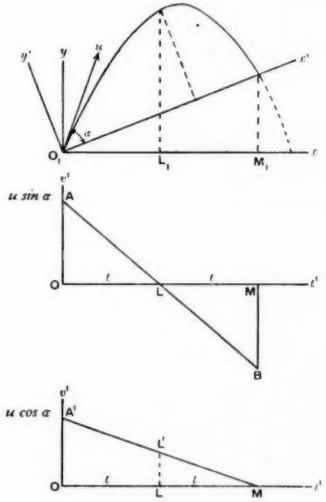


FIG. 2.

The (v, t) graph for motion perpendicular to the inclined plane is ALB.

Hence from the triangle AOL we have

$$u \sin \alpha = g \cos \beta \,.\, t. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The (v, t) graph for motion parallel to the inclined plane is $A'L'M$.
Hence from the triangle $A'OM$ we have

$$u \cos \alpha = g \sin \beta \,.\, 2t. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Dividing (1) by (2), $\tan \alpha = \frac{1}{2} \cot \beta$, whence $2 \tan \alpha = \cot \beta$.
Squaring (1) and (2) and adding,

$$u^2 = g^2t^2(\cos^2\beta + 4 \sin^2\beta) = g^2t^2(1 + 3 \sin^2\beta). \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{Area } \triangle A'OM = \text{Range} = u \cos \alpha \,.\, t.$$

Hence from (2) and (3), $\text{Range} = 2g \sin \beta \,.\, t^2 = 2 \dfrac{u^2}{g} \cdot \dfrac{\sin \beta}{1 + 3 \sin^2\beta}$.

3. A particle falls from a height h upon a fixed horizontal plane; if e is the coefficient of restitution, show that the whole distance described before the particle has finished bounding is $\dfrac{1+e^2}{1-e^2} h$ and that the whole time taken is

$$\sqrt{\left(\frac{2h}{g}\right)} \cdot \frac{1+e}{1-e}.$$

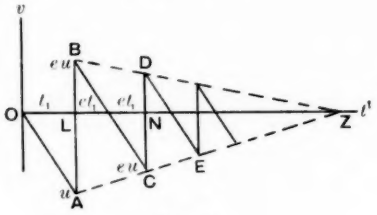


FIG. 3.

The (v, t) graph of the motion is $OABCDE \ldots$.
From similar triangles $t_2 = et_1$, $t_3 = e^2t_1$, etc.
The area $OAL = \frac{1}{2}gt_1^2 = h$, whence $t_1 = \sqrt{(2h/g)}$.
Also from similar triangles, BD and AC meet the time-axis at the same point Z.
Thus

$$LZ : LN = LA : (LA - NC).$$

Hence the total time is given by

$$OZ = OL + LZ = t_1 + \frac{u}{(1-e)u} \cdot 2et_1 = (1+e)t_1/(1-e).$$

The total distance is given by area OAL + area $ABZ = h + \frac{1}{2}(1+e)u \,.\, LZ$, whence the result.

4. A particle is projected from a point on a horizontal floor and after rebounding once from the floor strikes a vertical wall, when it is next moving horizontally. If it returns to the point of projection after rebounding twice from the floor, and e_1, e_2 are the coefficients of restitution at the floor and wall respectively, prove that $2 + e_1 = e_1e_2(1 + 2e_1 + 2e_1^2)$.
The (v, t) graph for vertical motion is $ABCDEFGH$.
As the triangles AOL, MCN, PEQ, RGS are similar,

$$t_1 = e_1t\,; \quad t_2 = e_1^2t\,; \quad t_3 = e_1^3t.$$

The (v, t) graph for horizontal motion is $A'N'N''T'$.
Also $NN'' = e_2 NN'$.
Distance of point of projection from the wall is given by

$$\text{Rect. } A'ONN' = \text{Rect. } NN''T'T.$$

Thus
$$2t + t_1 = e_2(t_1 + 2t_2 + 2t_3),$$

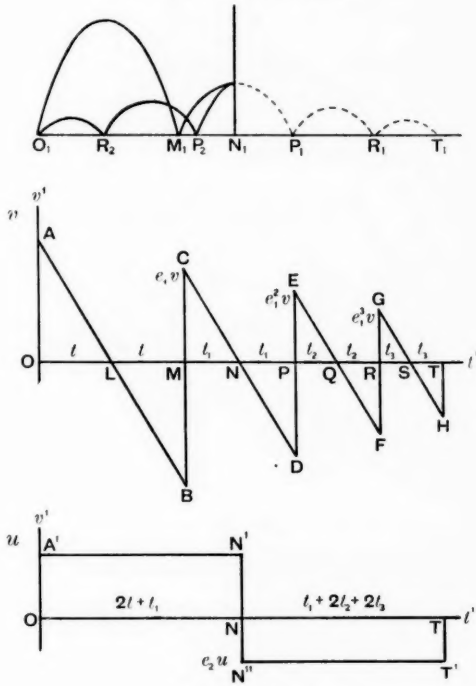


FIG. 4.

so that
$$(2 + e_1)t = e_2(e_1 + 2e_1^2 + 2e_1^3)t,$$

and hence
$$2 + e_1 = e_1 e_2(1 + 2e_1 + 2e_1^2).$$

5. A heavy particle is to be projected from a given point A to reach another point B on the same horizontal as A, after impacts on three walls of a squash court. Prove that there are two possible angles of projection which lie in the same vertical plane through A, provided the speed is great enough. Show that the lower trajectory takes the lesser time although the velocity on arrival is less. [Coefficients of restitution : e_1, e_2, e_3.] (Figs. 5*a*, 5*b*).

From geometry,
$$2t = t_1 + t_2 + t_3 + t_4.$$

The (v, t) graph for motion in z direction is CLD.

Thus
$$V \sin \alpha = gt. \quad \ldots\ldots\ldots\

The (v, t) graph for motion in y direction is $C_1F_1G_1D_1$.

Thus $$V \cos \alpha \sin \theta (t_1 + t_2) = b$$

and $$e_2 V \cos \alpha \sin \theta (t_3 + t_4) = k.$$

Hence $$e_2 V \cos \alpha \sin \theta \,.\, 2t = e_2 b + k. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

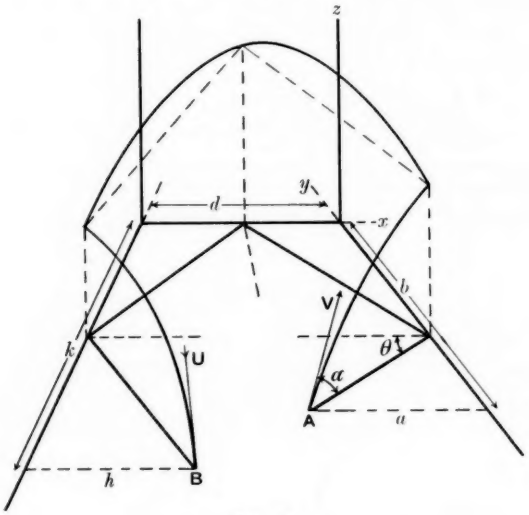


FIG. 5a.

The path from A to B consists of four parabolic arcs, divided by the points of contact with the three walls.

The (v, t) graph for motion in x direction is $C_2E_2F_2H_2J_2D_2$.

Thus $$V \cos \alpha \cos \theta \,.\, t_1 = a,$$

$$e_1 V \cos \alpha \cos \theta (t_2 + t_3) = d,$$

$$e_1 e_3 V \cos \alpha \cos \theta \,.\, t_4 = h.$$

Hence $$e_1 e_3 V \cos \alpha \cos \theta \,.\, 2t = a e_1 e_3 + d e_3 + h. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Dividing (2) by (3), $\tan \theta = \dfrac{e_1 e_3}{e_2} \dfrac{(e_2 b + k)}{(a e_1 e_3 + d e_3 + h)}$.

Hence we have one value of θ only.

Eliminating t between (1) and (2), $\sin 2\alpha = g(be_2 + k)/V^2 e_2 \sin \theta$.

Hence two values of α if $V^2 > (be_2 + k)g/e_2 \sin \theta$:

$$U^2 = V^2 [\sin^2 \alpha + \cos^2 \alpha (e_2^2 \sin^2 \theta + e_1^2 e_3^2 \cos^2 \theta)]$$

$$= V^2 [1 - \cos^2 \alpha (1 - e_2^2 \sin^2 \theta - e_1^2 e_3^2 \cos^2 \theta)].$$

Hence the smaller value of α gives the lesser value of U.

From (1), the smaller value of α gives the lesser value of $2t$.

6. Two masses m, M are connected by a light string passing over a smooth pulley, vertically above a smooth inelastic plane, M being held so as to prevent motion. If M is released and takes t seconds to reach the plane,

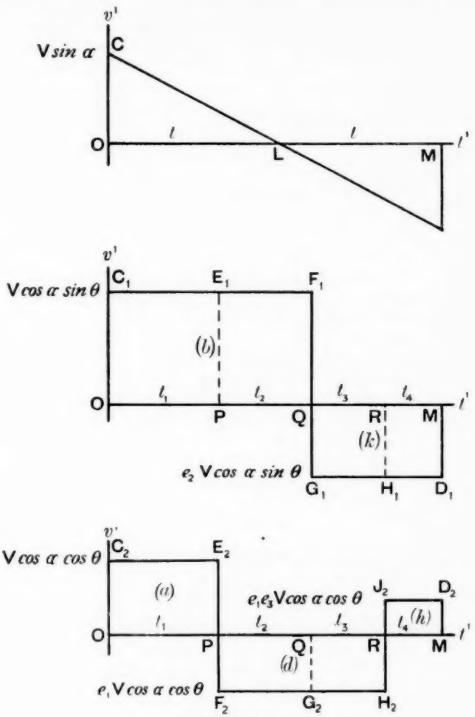


FIG. 5b.

show that the system will first be at rest instantaneously (with the string taut) after a time $3Mt/(M+m)$, and that the system will be finally at rest after a time $3t$. (Fig. 6).

The (v, t) graph for the mass M is $OAKPBN$ and that for the mass m is $OA'P'B'N$. From the triangles OAK, $OA'K$ we have $KA = KA' = ft$, where f is obtained from Newton's second law,

$$f = (M-m)g/(M+m).$$

From the triangle $A'KL$,

$$gt_1 = A'K = ft,$$

so that $t_1 = (M-m)t/(M+m)$.

When the jerk takes place,

$$M \,.\, BP = I,$$

$$m \,.\, PB' + I = m \,.\, PP',$$

where I is the impulsive tension in the string.

But $BP = PB'$.

Hence, adding, $(M+m) \,.\, PB = m \,.\, PP' = mft.$

But $BP = ft_2$.

Thus $t_2 = mt/(M+m)$.

The total time $T = t + 2t_1 + t_2 = 3Mt/(M+m)$.

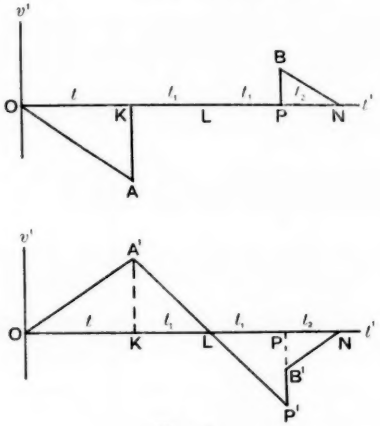


FIG. 6.

The continuation of the (v, t) graph is a repetition of the above with a scale factor :

$$BP/KA = m/(M+m).$$

Thus the time is $mT/(M+m)$, and the total time is then

$$T\left\{1 + \frac{m}{M+m} + \left(\frac{m}{M+m}\right)^2 + \ldots\right\} = T(M+m)/M,$$

since $m/(M+m) < 1$.

Hence the total time is

$$\frac{3Mt}{M+m} \cdot \frac{M+m}{M} = 3t.$$

J. N. B.

GLEANINGS FAR AND NEAR.

1608. An appeal to Britain's workpeople to increase their output by 10 per cent. on a five-day week, or by 2 per cent. more each day, was made by Mr. Isaacs, Minister of Labour, at Edmonton yesterday.—*The Times*, October 4, 1947. [Per Mr. R. H. Macmillan.]

1609. A new theorem : *Let $T_1, T_2, \ldots T_n$ denote n bounded numbers, n being a given positive integer. To prove that the sum $T_1 + T_2 + \ldots + T_n$ is unbounded.*

A limit is to be fixed for the number of women teachers in each area, though no restriction will be placed on the national total.—*Daily Telegraph*, March 22, 1948. [Per Dr. E. A. Maxwell.]

ON CARTOPHILY AND MOTOR CARS.

By H. J. Godwin.

In Vol. XXII, No. 251, of the *Mathematical Gazette*, there appeared an article by Dr. F. G. Maunsell in which were obtained the probabilities of collecting a set of r cigarette cards in r, $r+1$, $r+2$, ... draws: he obtained two expressions for the mean number of draws required. In Note 1345 (No. 253) a number of writers were cited as having demonstrated the identity of these expressions with a third simpler one, and one such demonstration was given in Note 1346. In Note 1968 (No. 295) Professor R. L. Goodstein referred again to the problem, giving yet another demonstration for one of the expressions. In this note I obtain an expression for any moment of the distribution of the number of draws and consider a related problem.

To generalise the problem somewhat, suppose that $(m+m')$ different kinds of cards are issued, all kinds being equally likely to occur at a draw, and that we are interested in m of these kinds (hereafter referred to as favourable). The probability of needing just n draws to get members of r favourable kinds is the chance of $(n-1)$ draws containing $(r-1)$ favourable kinds, multiplied by the chance of the nth draw being from another favourable kind. (It is perhaps worth noting that, save for $r=m$, it is not the chance of n draws containing r favourable kinds, less the chance of $(n-1)$ draws doing so, since we are unable to ensure the same set of r in the two cases. In fact, for $r<m$, the chance that n draws contain just r favourable kinds tends to zero as n tends to infinity.) Suppose that k of the draws are favourable. The chance of this is

$$\binom{n-1}{k}\frac{m^k m'^{n-k-1}}{(m+m')^{n-1}}.$$

The chance that k cards contain members of just $(r-1)$ kinds is

$$\binom{m}{r-1}\Delta^{r-1}(0^k)/m^k,$$

since the number of ways in which k draws can be made from $(r-1)$ kinds to contain at least one from each kind is

$$\Delta^{r-1}(0^k)=\sum_0^{r-1}(-)^s\binom{r-1}{s}(r-1-s)^k$$

(see, for example, Hardy and Wright, *Theory of Numbers*, Theorem 260). Summing the product of the two chances over $k=0, 1, \ldots (n-1)$, we get

$$\frac{\binom{m}{r-1}}{(m+m')^{n-1}}\sum_0^{r-1}(-)^s\binom{r-1}{s}(m'+r-1-s)^{n-1}.$$

The chance of the next draw being from another favourable kind is

$$(m-r+1)/(m+m').$$

Hence the chance of r favourable kinds being collected in just n draws is

$$\frac{m\binom{m-1}{r-1}}{(m+m')^n}\sum_0^{r-1}(-)^s\binom{r-1}{s}(m'+r-s-1)^{n-1}.$$

For $m'=0$, and $r=m$, this agrees with Dr. Maunsell's expression.

Although this is the exact expression for the required chance, it is too complicated to be of much use for calculation. In order to compute the chance of taking more or less than a given number of draws, or conversely the number

of draws of which there is a given chance of exceeding or falling short, we need to use some method of approximation. A standard method is to calculate the moments of the distribution (that is, the average values of n, n^2 ...) and to use, as an approximation to the distribution, that one of Pearson's system of curves with the same first four moments. (In some cases only the first two moments need be considered, and the properties of the normal curve of error used; if, however, as will appear here, the distribution is markedly skew, this is not possible.)

Instead of finding the moments directly, we find the average values of the polynomials n, $n(n+1)$, $n(n+1)(n+2)$, ... Since powers of n can be expressed as linear combinations of these, the moments can be expressed as linear combinations of their average values. For example, denoting average value by E we have for the third moment

$$E(n^3) = E\{n(n+1)(n+2)\} - 3E\{n(n+1)\} + E(n).$$

Now $E\{n(n+1)(n+2) \ldots (n+a-1)\}$

$$= \sum_{n=1}^{\infty} n(n+1) \ldots (n+a-1) \frac{m\binom{m-1}{r-1}}{(m+m')^n} \sum_{s=0}^{r-1} (-)^s \binom{r-1}{s} (m'+r-1-s)^{n-1}$$

$$= \sum_{n=1}^{\infty} a! \binom{n+a-1}{a} \frac{m\binom{m-1}{r-1}}{(m+m')^n} \sum_{s=0}^{r-1} (-)^s \binom{r-1}{s} (m'+r-1-s)^{n-1}$$

$$= \sum_{0}^{r-1} (-)^s \binom{r-1}{s} \frac{m\binom{m-1}{r-1}}{(m+m')} \left\{1 - \frac{m'+r-s-1}{m+m'}\right\}^{-a-1} . a!,$$

(on using the binomial theorem for a negative integral index),

$$= \sum_{0}^{r-1} (-)^s \binom{r-1}{s} \frac{m(m+m')^a}{(m-r+s+1)^{a+1}} \binom{m-1}{r-1} a! \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

This gives yet another expression for the first moment in Dr. Maunsell's case, namely,

$$m^2 \sum_{0}^{m-1} \binom{m-1}{s} \frac{(-)^s}{(s+1)^2}.$$

We now express these expectations of polynomials in terms of sums of powers of the reciprocals of the integers between $m-r+1$ and m. In order to do this we denote by $f(r, a)$ the expression

$$\sum_{0}^{r-1} (-)^s \binom{r-1}{s} \frac{m\binom{m-1}{r-1}}{(m-r+s+1)^{a+1}}$$

so that the expression in (i) is $(m+m')^a(a!)f(r, a)$. Then

$$f(r, a) - f(r-1, a) = m \sum_{0}^{r-1} \left\{ \frac{(-)^s \binom{r-1}{s}\binom{m-1}{r-1}}{(m-r+s+1)^{a+1}} + \frac{(-)^s \binom{r-2}{s-1}\binom{m-1}{r-2}}{(m-r+s+1)^{a+1}} \right\}$$

$$= m \sum_{0}^{r-1} \frac{(-)^s (m-1)!(m-r+1+s)}{(m-r+s+1)^{a+1}(m-r+1)!\, s!\, (r-1-s)!}$$

$$= f(r, a-1)/(m-r+1).$$

Also $f(r, -1) = 0$, so that $f(r, 0) = f(1, 0) = 1$ and $f(1, a) = 1/m^a$. Hence $f(r, a)$ is the sum of the homogeneous products of weight a formed from the quan-

tities $1/m$, $1/(m-1)$, ... , $1/(m-r+1)$. There are several ways of arriving at the expression of $f(r, a)$ in terms of S_i, where $S_i = \sum_{\nu=m-r+1}^{m} 1/\nu^i$; one is to note that $f(r, a)$ is the coefficient of t^a in the formal expansion of

$$\begin{aligned}
&\left(1+\frac{t}{m-r+1}+\frac{t^2}{(m-r+1)^2}+\dots\right)\dots\left(1+\frac{t}{m}+\frac{t^2}{m^2}+\dots\right)\\
&=\left(1-\frac{t}{m-r+1}\right)^{-1}\dots\left(1-\frac{t}{m}\right)^{-1}\\
&=\exp\left\{-\log\left(1-\frac{t}{m-r+1}\right)-\dots-\log\left(1-\frac{t}{m}\right)\right\}\\
&=\exp\left\{\frac{t}{m-r+1}+\frac{t^2}{2(m-r+1)^2}+\dots+\frac{t}{m}+\frac{t^2}{2m^2}+\dots\right\}\\
&=\exp\{S_1t+\tfrac{1}{2}S_2t^2+\dots\}.
\end{aligned}$$

Now the coefficient of t^a in this is the sum

$$\Sigma\frac{S_i^{\alpha}S_j^{\beta}\dots}{\alpha!\,\beta!\dots i^{\alpha}j^{\beta}\dots},$$

over all sets of positive integers $i, j, \dots \alpha, \beta, \dots$ for which $\alpha i+\beta j+\dots=a$.

By means of the above chain of relations we have now expressed the moments of the distribution of n in terms of the S_i. For example, first moment $=(m+m')S_1$, second moment $=(m+m')^2(S_2+S_1{}^2)-(m+m')S_1$.

In the case where $m'=0$, $r=m=50$, we find that, while the average value of n is 225, there is a chance of 1 per cent. of collecting the set in less than 130 draws, and a 1 per cent. chance of needing more than 430 draws (the last two values being given to 2 significant figures).

A related problem is that in which the order in which categories are obtained is important. A practical example may be found in the game in which one seeks a motor-car with registration number 1, then one with number 2, and so on. We again assume that all categories occur with equal frequency (though in the given illustration this is not realised, owing to the predominance of low numbers in uncompleted sets of letters): we may take $m'=0$, since all categories other than that immediately being sought are unfavourable.

The chance of getting r categories in just n draws is

$$\binom{n-1}{r-1}(m-1)^{n-r}/m^n.$$

For $(r-1)$ of the first $(n-1)$ draws, and the nth draw, have been favourable, with probability $1/m$, and the remainder unfavourable, with probability $(m-1)/m$. The average value of $n(n+1)\dots(n+a-1)$ is

$$\begin{aligned}
&\sum_{n=r}^{\infty} n(n+1)\dots(n+a-1)\binom{n-1}{r-1}(m-1)^{n-r}/m^n\\
&=\frac{(r+a-1)!}{m^r(r-1)!}\sum_{n=r}^{\infty}\binom{n+a-1}{r+a-1}\left(\frac{m-1}{m}\right)^{n-r}\\
&=(r+a-1)!\,m^a/(r-1)!,
\end{aligned}$$

again using the binomial theorem for a negative integral index. Hence the first moment of n is rm, and the second moment of n is $rm\{m(r+1)-1\}$. If $r=m=999$, then we find that the average value of n is 998,000, that there is a 1 per cent. chance of n being less than 925,000 and a 1 per cent. chance of n being greater than 1,071,000 (with values to the nearest thousand).

H. J. G.

THE EULER-MACLAURIN FORMULA.

By H. G. Forder.

The most surprising thing about this formula is its use of the Bernoulli numbers, and it is natural to ask why they appear. The answer is that it is not the B's which insist on entry, but the numbers $A_r = B_r/r!$. We use the notation where $B_1 = -\frac{1}{2}$, all other B with odd subscripts vanish, and those with even subscripts are alternately positive and negative. ($B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ...). We shall often ignore the fact that A_3, A_5, ... vanish, thereby refusing to complicate formulae unnecessarily.

1. Writing $\Delta f(a)$ for $f(a+h) - f(a)$, a well-known form of Taylor's theorem runs as follows :

$$\Delta f(a) = \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) \qquad + \int_0^h \frac{x^n}{n!}P(x)\,dx,$$

where $P(x) = f^{(n+1)}(a+h-x)$.

Put f' for f, $n-1$ for n ; then

$$\Delta f'(a) = \qquad \frac{h}{1!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n)}(a) + \int_0^h \frac{x^{n-1}}{(n-1)!}P(x)\,dx,$$

So proceed for $\Delta f''(a)$, $\Delta f'''(a)$, ... , until

$$\Delta f^{(n-1)}(a) = \qquad \frac{h}{1!}f^{(n)}(a) \qquad + \int_0^h \frac{x}{1!}P(x)\,dx.$$

We multiply these equations in turn by 1, $A_1 h$, $A_2 h^2$, ... , $A_{n-1}h^{n-1}$ and add, choosing the A so that the coefficients of f'', f''', ... $f^{(n)}$ in the result vanish. Thus the A have to satisfy

$$A_1 + \frac{1}{2!} = 0, \quad A_2 + \frac{A_1}{2!} + \frac{1}{3!} = 0, \quad A_3 + \frac{A_2}{2!} + \frac{A_1}{3!} + \frac{1}{4!} = 0, \quad \ldots\ldots\ldots\ldots(1)$$

These enable us to calculate the A in turn.

The addition mentioned gives

$$\Delta f(a) + A_1 h\,\Delta f'(a) + A_2 h^2 \Delta f''(a) + \ldots + A_{n-1}h^{n-1}\Delta f^{(n-1)}(a) = hf'(a) - R_n, \quad \ldots(2)$$

where $$R_n = -\int_0^h \left\{\frac{x^n}{n!} + A_1\frac{x^{n-1}h}{(n-1)!} + A_2\frac{x^{n-2}h^2}{(n-2)!} + \ldots + A_{n-1}\frac{xh^{n-1}}{1!}\right\}P(x)\,dx.$$

Putting $y = x/h$ and then writing x again for y, we have

$$R_n = -h^{n+1}\int_0^1 \left\{\frac{x^n}{n!} + A_1\frac{x^{n-1}}{(n-1)!} + A_2\frac{x^{n-2}}{(n-2)!} + \ldots + A_{n-1}\frac{x}{1!}\right\}P(hx)\,dx$$

$$= -h^{n+1}\int_0^1 \phi_n(x) f^{(n+1)}\{a + h(1-x)\}\,dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $$\phi_n(x) = \frac{x^n}{n!} + A_1\frac{x^{n-1}}{(n-1)!} + A_2\frac{x^{n-2}}{(n-2)!} + \ldots + A_{n-1}\frac{x}{1!}. \quad \ldots\ldots\ldots\ldots(4)$$

We observe that $$\phi_n(0) = 0, \quad \phi_n(1) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\phi'_{n+1}(x) = \phi_n(x) + A_n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

2. The relation of the A and the ϕ to the Bernoulli numbers and polynomials is clear :

If we expand $x/(e^x - 1)$ *formally* as $A_0 + A_1 x + A_2 x^2 + \ldots$

then $$(A_0 + A_1 x + A_2 x^2 + \ldots)(x + x^2/2! + x^3/3! + \ldots) = x$$

gives the relations (1) between the A, and since

$$\frac{x}{e^x-1}+x=\frac{xe^x}{e^x-1}=\frac{-x}{e^{-x}-1}=A_0-A_1x+A_2x^2-\dots,$$

we find, comparing with the first expression, $A_3=A_5=\dots=0$, a fact we shall usually disregard.

2·1. The coefficient of t^n in the *formal* expansion of

$$t(e^{xt}-1)/(e^t-1)=(A_0+A_1t+A_2t^2+\dots)(xt+x^2t^2/2!+x^3t^3/3!+\dots)$$

is clearly $\phi_n(x)$.

3. Writing f instead of f', and putting $a+h$, $a+2h$, $\dots$ $a+(s-1)h$ for a in (2) and adding, we get the usual sum-formula :

$$h\sum_{t=0}^{s-1}f(a+th)=\int_a^{a+sh}f(x)dx+\sum_{r=1}^{n-1}A_rh^r\{f^{(r-1)}(a+sh)-f^{(r-1)}(a)\}+R_n',$$

where

$$R_n'=-h^{n+1}\sum_{t=1}^{s}\int_0^1\phi(x)f^{(n)}\{a+h(t-x)\}dx.$$

4. The most interesting case is when we have an infinite sum and then we must consider the remainder after a finite number of terms. This remainder is a finite set of terms of form (3), and it is this form we now consider. We shall show later :

A : $\phi_{2m}(x)$ has a constant sign when $0<x<1$;

C : $|\phi_{2m}(x)|<2|A_{2m}|$ in the same range of x.

4·1. Assuming A we have, by the mean-value theorem,

$$\int_0^1\phi_{2m}(x)f^{(2m+1)}\{a+h(1-x)\}dx=f^{(2m+1)}(a+h\theta)\int_0^1\phi_{2m}(x)dx$$
$$=-A_{2m}f^{(2m+1)}(a+h\theta),\quad(0<\theta<1)$$

since by (5), (6),

$$\int_0^1\phi_{2m}(x)dx+A_{2m}=\Big[\phi_{2m+1}(x)\Big]_0^1=0.$$

Hence, by (3), for any sufficiently continuous function

$$R_{2m}=A_{2m}h^{2m+1}f^{(2m+1)}(a+h\theta).$$

If we assume that $f^{(2m+1)}\{a+h(1-x)\}$ has constant sign when $0<x<1$, we can replace this, if we wish, by

$$R_{2m}=-h^{2m+1}\phi_{2m}(\theta)\int_0^1f^{(2m+1)}\{a+h(1-x)\}dx,\quad(0<\theta<1)$$
$$=h^{2m}\phi_{2m}(\theta)\{f^{(2m)}(a+h)-f^{(2m)}(a)\},$$

which gives, if we assume C,

$$|R_{2m}|<2|A_{2m}|\,h^{2m}\Delta f^{(2m)}(a).$$

5. The values of the first A are found to be

$$A_1=-\tfrac{1}{2},\quad A_2=\tfrac{1}{12},\quad A_4=-\tfrac{1}{720},\quad A_6=\tfrac{1}{30240},$$
$$A_8=-\tfrac{1}{1209600},\quad A_{10}=\tfrac{1}{47900160}.$$

5·1. Take, for example, $f(x)=\log x$, $f^{(n)}(x)=(-)^{n-1}(n-1)!\,x^{-n}$, and we find

$$\sum_{t=0}^{s-1}\log(a+t)=\int_a^{a+s}\log x\,dx-\tfrac{1}{2}\{\log(a+s)-\log a\}$$
$$+\sum_{2}^{n-1}A_r(r-2)!\left\{\frac{1}{(a+s)^{r-1}}-\frac{1}{a^{r-1}}\right\}+R_n'.$$

Putting $a = 1$, we get

$$\log(s!) = (s+\tfrac{1}{2})\log(s+1) - s + \sum_{2}^{n-1} A_r(r-2)!\left\{\frac{1}{(1+s)^{r-1}} - 1\right\} + R'_n,$$

or, writing $s-1$ for s and taking some terms with remainder,

$$\log(s!) = C + (s+\tfrac{1}{2})\log s - s + \sum_{2}^{n-1}\frac{A_r(r-2)!}{s^{r-1}} + R_n,$$

where C is a constant which, using Wallis' product for π, can be shown to be $\frac{1}{2}\log(2\pi)$. This gives Stirling's approximation to $\log(s!)$.

The residual series begins with

$$\frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \frac{1}{1188s^9} - \dots .$$

The series does not converge.

The coefficients $C_{2m} = A_{2m}(2m-2)!$ *increase rapidly* after C_8; for we shall see that A_{2m} is roughly $2(2\pi)^{-2m}$, and thus

$$C_{2m+2}/C_{2m} \text{ is approximately } m^2/\pi^2.$$

6. *Bernoulli polynomials.* We defined $\phi_n(x)$ by the formal expansion

$$t(e^{xt}-1)/(e^t-1) = \Sigma\phi_n(x)t^n.$$

Thus $\quad \phi_1(x) = x, \quad \phi_2(x) = \frac{1}{2}(x^2 - x), \quad \phi_3(x) = \frac{1}{12}(2x^3 - 3x^2 + x).$

6·1. Since $\phi_n(x+1) - \phi_n(x)$ is the coefficient of t^n in

$$t\{e^{(1+x)t} - e^{xt}\}/(e^t-1) = te^{xt},$$

we get $\qquad \phi_n(x+1) - \phi_n(x) = x^{n-1}/(n-1)!,$

whence again $\qquad \phi_n(1) = 0,$

and further, $\quad \sum_{t=0}^{s-1} t^{n-1} = 1 + 2^{n-1} + 3^{n-1} + \dots + (s-1)^{n-1} = (n-1)!\,\phi_n(s).$

6·2. $\phi_n(1-x) = (-)^n\phi_n(x), \quad (n = 2, 3, \dots).$

For
$$\begin{aligned}\Sigma\phi_n(1-x)(-t)^n &= -t\{e^{(1-x)(-t)} - 1\}/(e^{-t}-1)\\ &= t(e^{xt} - e^t)/(e^t - 1)\\ &= t(e^{xt}-1)/(e^t-1) - t\\ &= \Sigma\phi_n(x)t^n - t.\end{aligned}$$

6·3. $\phi_n(\frac{1}{2})$ is the coefficient of t^n in the expansion of

$$t\frac{e^{t/2}-1}{e^t-1} = t\left\{\frac{e^{t/2}+1}{e^t-1} - \frac{2}{e^t-1}\right\} = 2\left\{\frac{t/2}{e^{t/2}-1} - \frac{t}{e^t-1}\right\}.$$

Hence the coefficient is $\quad 2(A_n 2^{-n} - A_n),$

and hence, if n is odd, $\quad \phi_n(\frac{1}{2}) = 0,$

if n is even, $\quad \phi_n(\frac{1}{2}) = -2A_n(1 - 2^{-n}).$

7. We can now show :

A : $\phi_{2m}(x)$ has no root between 0 and 1, and thus has constant sign there :

B : $\phi_{2m+1}(x)$ has $x = \frac{1}{2}$ as the sole root between 0 and 1.

For, as these can be verified when $m = 1$, we may assume them true for $m \leqslant \mu$, and show them for $m = \mu + 1$.

By (6), $\qquad \phi'_{2\mu+2}(x) = \phi_{2\mu+1}(x)$, since $A_{2\mu+1} = 0$.

Thus $\phi_{2\mu+2}$ has a turning value at $x = \frac{1}{2}$ only, and as it vanishes at $x = 0, 1$,

it cannot vanish between these values, for if it did, it would have at least two turning values. Hence A for $m = \mu + 1$. Next

$$\phi'_{2\mu+3}(x) = \phi_{2\mu+2}(x) + A_{2\mu+2},$$

and the right-hand side has at most one root between 0 and $\frac{1}{2}$, and hence $\phi_{2\mu+3}(x)$ has at most one turning value there, and as it vanishes at $x = 0, \frac{1}{2}$, it cannot vanish between these values, and hence by 6·2, neither can it vanish between $\frac{1}{2}$ and 1. Hence B for $m = \mu + 1$.

7·1. We now deduce C, assumed in 4·1. The only turning value of $\phi_{2m}(x)$, between 0 and 1, is at $x = \frac{1}{2}$. Hence in that range, by 6·3,

$$|\phi_{2m}(x)| \leqslant |\phi_{2m}(\tfrac{1}{2})| < 2|A_{2m}|.$$

7·2. If $m > 1$, then $\phi'_{2m}(x) = 0$ at $x = 0$, 1, by (5) and (6). Hence $\phi_{2m}(x)$ has a factor $x^2(x-1)^2$.

7·3. $\phi_{2m-1}(x)$ and $\phi_{2m+1}(x)$ have contrary signs for the same value of x between 0 and 1.

For, since ϕ_{2m+1} vanishes at $x = 0, \frac{1}{2}$, its derivative vanishes at some point x_1 between. Thence

$$\phi_{2m+1}(x_1 + h) = \phi_{2m+1}(x_1) + \tfrac{1}{2}h^2\phi''_{2m+1}(x_1 + \theta h). \quad (0 < \theta < 1)$$

Put $h = -x_1$; thence, using (6),

$$\phi_{2m+1}(x_1) = -\tfrac{1}{2}x_1^2\phi''_{2m+1}(\theta' x_1) = -\tfrac{1}{2}x_1^2\phi_{2m-1}(\theta' x_1), \quad (0 < \theta' < 1)$$

Hence the result by 7B, in the range 0 to $\frac{1}{2}$, and thence by 6·2 in the range $\frac{1}{2}$ to 1.

7·4. $\phi_{2m-2}(x)$ and $\phi_{2m}(x)$ have contrary signs for the same value of x between 0 and 1.

For $\phi_{2m}(x)$ has a constant sign there, namely, that of its derivative $\phi_{2m-1}(x)$ near $x = 0$. Thus 7·3 gives the result.

7·5. Since $\phi_{2m}(\frac{1}{2})$ and A_{2m} have opposite signs, we thus have $A_2 > 0$, $A_4 < 0$, $A_6 > 0$, ... All these A are non-zero. No Bernoulli number with even subscript can vanish.

8. Define

$$\Phi_p(x) = \sum_1^\infty \frac{\sin n\theta}{n^p} \quad \text{for } p \text{ odd},$$

$$\Phi_p(x) = \sum_1^\infty \frac{\cos n\theta}{n^p} \quad \text{for } p \text{ even},$$

where $\theta = 2\pi x$. When $p > 1$, these series converge uniformly in any interval, and hence

$$\Phi'_{p+1}(x) = (-)^p 2\pi \Phi_p(x).$$

We shall show that, if $m > 1$, we have for suitable constant λ_m,

$$\phi_m(x) = \lambda_m\{\Phi_m(x) - \Phi_m(0)\}, \text{ when } 0 \leqslant x \leqslant 1.$$

For, first assume this for $m = r$; then

$$\begin{aligned}\phi_{r+1}(x) &= \int_0^x \phi'_{r+1}(x)\,dx = \int_0^x \{\phi_r(x) + A_r\}dx \\ &= \lambda_r \int_0^x \{\Phi_r(x) + \text{const.}\}dx \\ &= (-)^r(1/2\pi) \cdot \lambda_r \Phi_{r+1}(x) + Cx + D,\end{aligned}$$

where C, D are constants. Since $\phi_{r+1}(x)$ vanishes at $x = 0$, 1, we find $C = 0$, and so

$$\phi_{r+1}(x) = (-)^r(1/2\pi) \cdot \lambda_r\{\Phi_{r+1}(x) - \Phi_{r+1}(0)\}.$$

Hence if the theorem holds when $r=2$, it follows by induction generally,

$$\lambda_m = (-)^{m-1}\lambda_{m-1}/2\pi = (-)^{(m-1)+(m-2)+\cdots+2}\lambda_2/(2\pi)^{m-2}$$
$$= (-)^{(m+1)(m+2)/2}\lambda_2/(2\pi)^{m-2}.$$

To show the formula for $r=2$, we prove, for the sake of completeness, some extremely well-known results.

Put
$$F(\theta) = \sum_1^m \frac{\sin r\theta}{r} + \tfrac{1}{2}\theta + \frac{\cos(m+\frac{1}{2})\theta}{(2m+1)\sin\frac{1}{2}\theta}.$$

This is defined between 0 and 2π, and indeed at any point not a multiple of 2π.

Using
$$\tfrac{1}{2} + \cos\theta + \cos 2\theta + \ldots + \cos m\theta = \sin(m+\tfrac{1}{2})\theta/2\sin\tfrac{1}{2}\theta,$$

we find, if $0<\theta<2\pi$,

$$F'(\theta) = -\cos\tfrac{1}{2}\theta\cos(m+\tfrac{1}{2})\theta/2(2m+1)\sin^2\tfrac{1}{2}\theta.$$

The mean value theorem gives

$$F(\theta) - F(\pi) = F'(\xi)\,.\,(\theta-\pi),$$

where ξ is some value between θ and π. Now $F'(\xi)\to 0$ as $m\to\infty$, and hence

$$\sum_1^\infty \frac{\sin r\theta}{r} = \tfrac{1}{2}(\pi-\theta), \quad \text{if } 0<\theta<2\pi.$$

The series is uniformly convergent in any interval contained in this range. Hence by integration

$$\sum_1^\infty \frac{\cos m\theta - 1}{m^2} = \tfrac{1}{4}\theta^2 - \tfrac{1}{2}\pi\theta = \pi^2(x^2-x) = \pi^2\,.\,2\phi_2(x),$$

whence the required result with $\lambda_2 = 1/2\pi^2$.

Hence
$$\lambda_m = (-)^{(m+1)(m+2)/2}2/(2\pi)^m.$$

8·1. We have

$$\Phi_m(x) - \Phi_m(0) = (-)^{(m+1)(m+2)/2}\,.\,\tfrac{1}{2}(2\pi)^m\phi_m(x).$$

Taking $m=2r$, even, we have

$$\sum_{n=1}^\infty \frac{1-\cos n\theta}{n^{2r}} = (-)^r\,.\,\tfrac{1}{2}\,.\,(2\pi)^{2r}\phi_{2r}(x).$$

Put $x=\frac{1}{2}$, hence $\theta=\pi$.

$$2\sum_{n=1}^\infty (2n+1)^{-2r} = (-)^{r+1}(2\pi)^{2r}A_{2r}(1-2^{-2r}).$$

Define
$$S_{2r} = \sum_{n=1}^\infty n^{-2r} = \sum_1^\infty (2n+1)^{-2r} + 2^{-2r}S_{2r}.$$

Then
$$S_{2r} = \left\{\sum_1^\infty (2n+1)^{-2r}\right\} \Big/ (1-2^{-2r}) = (-)^{r+1}\,.\,\tfrac{1}{2}(2\pi)^{2r}A_{2r}.$$

8·2. This gives us the radius of convergence of the expansion of

$$x/(e^x-1)$$

as
$$\lim\left\{\frac{(2\pi)^{2r}}{2S_{2r}}\right\}^{1/2r} = 2\pi.$$

My thanks are due to Mr. B. I. Hayman for checking this note. H. G. F.

ON THE MULTIPLICATION OF COMPLEX NUMBERS.

By R. A. Newing.

In the development of the theory of complex numbers, it is important to give a definition of them dependent only upon real numbers. In the usual algebraic treatments the product is postulated in the form

$$(x, y)(x', y') = (xx' - yy', xy' + x'y),$$

or obtained from a matrix representation. The object of the present note is to show by elementary methods that the assumption that the multiplication of complex numbers is distributive, associative, and such that the vanishing of a product implies the vanishing of at least one factor, is sufficient to define the product. This is indeed merely a special case of results well known in the theory of linear algebras.

A complex number $\mathbf{q}$ may be defined in the usual way to be an ordered pair of real numbers (q_1, q_2). Two complex numbers $\mathbf{q}$, $\mathbf{q}'$ are equal if, and only if, $q_1 = q_1'$ and $q_2 = q_2'$; in particular $\mathbf{q} = \mathbf{0}$, where $\mathbf{0}$ is the complex zero $(0, 0)$, implies $q_1 = q_2 = 0$. The sum $\mathbf{q} + \mathbf{q}'$ is defined to be the complex number $(q_1 + q_1', q_2 + q_2')$, and multiplication by a real number n is defined to be such that $n\mathbf{q} = \mathbf{q}n = (nq_1, nq_2)$. The subtraction of $\mathbf{q}'$ from $\mathbf{q}$ is a permissible operation resulting in the complex number $\mathbf{q}''$, for $\mathbf{q}''$ exists such that $\mathbf{q}' + \mathbf{q}'' = \mathbf{q}$ since this implies $q_1' + q_1'' = q_1$, $q_2' + q_2'' = q_2$, and therefore

$$\mathbf{q}'' = (q_1 - q_1', q_2 - q_2') = \mathbf{q} - \mathbf{q}'.$$

Theorem 1. *If* $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ *are two given non-zero complex numbers such that there is no real number* n *for which* $\boldsymbol{\alpha} = n\boldsymbol{\beta}$ *(i.e. if* $\alpha_1\beta_2 \neq \alpha_2\beta_1$*), then any complex number* $\mathbf{z}$ *may be expressed as* $\mathbf{z} = \xi\boldsymbol{\alpha} + \eta\boldsymbol{\beta}$ *where* ξ, η *are real numbers which may be uniquely determined.*

This result follows at once, since $z_1 = \xi\alpha_1 + \eta\beta_1$, $z_2 = \xi\alpha_2 + \eta\beta_2$, and these simultaneous equations in ξ, η possess a unique solution if $\alpha_1\beta_2 \neq \alpha_2\beta_1$. It also follows that if $\mathbf{z}' = \xi'\boldsymbol{\alpha} + \eta'\boldsymbol{\beta}$, then $\mathbf{z} = \mathbf{z}'$ only if $\xi = \xi'$ and $\eta = \eta'$; in particular, $\mathbf{z} = \mathbf{0}$ if, and only if, $\xi = \eta = 0$.

The product of complex numbers is now defined to be such that: (i) the product of two complex numbers is itself a complex number, (ii) the distributive and (iii) the associative laws are satisfied, *i.e.*

$$(\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D}) = \mathbf{AC} + \mathbf{AD} + \mathbf{BC} + \mathbf{BD}, \quad \text{and} \quad (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}),$$

(iv) the vanishing of the product of two complex numbers implies the vanishing of at least one of the factors, *i.e.* $\mathbf{zz}' = \mathbf{0}$ if, and only if, $\mathbf{z} = \mathbf{0}$ or $\mathbf{z}' = \mathbf{0}$.

It follows from (iv) that if $\mathbf{A} = \mathbf{B}$, then $\mathbf{AC} = \mathbf{BC}$ for any $\mathbf{C}$,

$$\text{for} \quad \mathbf{0} = (\mathbf{A} - \mathbf{B})\mathbf{C} = \mathbf{AC} - \mathbf{BC}, \quad \text{using (ii).}$$

Let any two complex numbers $\mathbf{z}$, $\mathbf{z}'$ be expressed, as in theorem 1, in the form $\xi\boldsymbol{\alpha} + \eta\boldsymbol{\beta}$, $\xi'\boldsymbol{\alpha} + \eta'\boldsymbol{\beta}$, then from (ii)

$$\mathbf{zz}' = \xi\xi'\boldsymbol{\alpha}^2 + \eta\eta'\boldsymbol{\beta}^2 + \xi\eta'\boldsymbol{\alpha}\boldsymbol{\beta} + \xi'\eta\boldsymbol{\beta}\boldsymbol{\alpha}.$$

But from (i), $\mathbf{zz}'$ must be of the form $m\boldsymbol{\alpha} + n\boldsymbol{\beta}$, where m and n are real numbers. It must therefore be possible to find real numbers a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 such that

$$\boldsymbol{\alpha}^2 = a_1\boldsymbol{\alpha} + a_2\boldsymbol{\beta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\boldsymbol{\beta}^2 = b_1\boldsymbol{\alpha} + b_2\boldsymbol{\beta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\boldsymbol{\alpha}\boldsymbol{\beta} = c_1\boldsymbol{\alpha} + c_2\boldsymbol{\beta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\boldsymbol{\beta}\boldsymbol{\alpha} = d_1\boldsymbol{\alpha} + d_2\boldsymbol{\beta}.$$

Theorem 2. *Multiplication satisfies the commutative law.*

To prove this, note that from (iii) $\boldsymbol{\alpha}^2\boldsymbol{\alpha} = \boldsymbol{\alpha}\boldsymbol{\alpha}^2$, and therefore

$$a_1\boldsymbol{\alpha}^2 + a_2\boldsymbol{\beta}\boldsymbol{\alpha} = a_1\boldsymbol{\alpha}^2 + a_2\boldsymbol{\alpha}\boldsymbol{\beta},$$

which gives $a_2\boldsymbol{\beta}\boldsymbol{\alpha} = a_2\boldsymbol{\alpha}\boldsymbol{\beta}$.

Similarly $\boldsymbol{\beta}^2\boldsymbol{\beta} = \boldsymbol{\beta}\boldsymbol{\beta}^2$ gives $b_1\boldsymbol{\alpha}\boldsymbol{\beta} = b_1\boldsymbol{\beta}\boldsymbol{\alpha}$. Thus $\boldsymbol{\alpha}\boldsymbol{\beta} = \boldsymbol{\beta}\boldsymbol{\alpha}$ unless a_2 and b_1 are both zero. But if $a_2 = b_1 = 0$, then $\boldsymbol{\alpha}^2 = a_1\boldsymbol{\alpha}$, $\boldsymbol{\beta}^2 = b_2\boldsymbol{\beta}$, and therefore $\boldsymbol{\alpha}(b_2\boldsymbol{\alpha} - a_1\boldsymbol{\beta})\boldsymbol{\beta} = \mathbf{0}$; but this is not possible since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both non-zero, and $b_2\boldsymbol{\alpha} - a_1\boldsymbol{\beta}$ cannot vanish as this would imply $\alpha_1\beta_2 = \alpha_2\beta_1$. It therefore follows that $\boldsymbol{\alpha}\boldsymbol{\beta} = \boldsymbol{\beta}\boldsymbol{\alpha}$, and therefore $\mathbf{z}\mathbf{z}' = \mathbf{z}'\mathbf{z}$ for any complex numbers $\mathbf{z}$ and $\mathbf{z}'$.

Theorem 3. *If $\mathbf{z} \neq \mathbf{0}$, the division of $\mathbf{z}'$ by $\mathbf{z}$ is a permissible operation resulting in a unique complex number $\mathbf{z}''$.*

Condition (iv) requires that the equations

$$\xi'(\xi a_1 + \eta c_1) + \eta'(\xi c_1 + \eta b_1) = 0,$$
$$\xi'(\xi a_2 + \eta c_2) + \eta'(\xi c_2 + \eta b_2) = 0,$$

should imply $\xi' = \eta' = 0$ for any ξ, η if $\mathbf{z} \neq \mathbf{0}$. This will be so if

$$\xi^2(a_1c_2 - a_2c_1) + \eta^2(b_2c_1 - b_1c_2) + \xi\eta(a_1b_2 - a_2b_1) \neq 0,$$

and this requires that

$$\left.\begin{aligned} a_1c_2 \neq a_2c_1, \quad b_2c_1 \neq b_1c_2, \\ 4(a_1c_2 - a_2c_1)(b_2c_1 - b_1c_2) > (a_1b_2 - a_2b_1)^2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now, to establish the theorem, it is merely necessary to show that, for given $\mathbf{z}$, $\mathbf{z}'$ a unique $\mathbf{z}''$ may be found such that $\mathbf{z}\mathbf{z}'' = \mathbf{z}'$, *i.e.* it must be possible to solve the simultaneous equations in ξ'', η'' :

$$\xi''(\xi a_1 + \eta c_1) + \eta''(\xi c_1 + \eta b_1) = \xi',$$
$$\xi''(\xi a_2 + \eta c_2) + \eta''(\xi c_2 + \eta b_2) = \eta'.$$

This will be so on account of (4).

Theorem 4. *There is one and only one complex number $\mathbf{I}$ such that $\mathbf{I}\mathbf{z} = \mathbf{z}$.*

Since $\mathbf{z}''$ may be found such that $\mathbf{z}''\mathbf{z} = \mathbf{z}'$ for any non-zero $\mathbf{z}$ and any $\mathbf{z}'$, then $\mathbf{I}$ may be found such that $\mathbf{I}\mathbf{z} = \mathbf{z}$. The number $\mathbf{I}$ so found is such that $\mathbf{I}\mathbf{q} = \mathbf{q}$, where $\mathbf{q}$ is any other complex number. For, given $\mathbf{q}$ and $\mathbf{z}$, $\mathbf{f}$ may be found such that $\mathbf{q} = \mathbf{z}\mathbf{f}$ and therefore $\mathbf{I}\mathbf{q} = \mathbf{I}(\mathbf{z}\mathbf{f}) = (\mathbf{I}\mathbf{z})\mathbf{f} = \mathbf{z}\mathbf{f} = \mathbf{q}$. There can only be one such number $\mathbf{I}$; for let $\mathbf{I}'$ be another number such that $\mathbf{I}'\mathbf{z} = \mathbf{z}$, then $\mathbf{I}'\mathbf{q} = \mathbf{q}$ for any $\mathbf{q}$, and therefore in particular $\mathbf{I}'\mathbf{I} = \mathbf{I}$. But $\mathbf{I}\mathbf{q} = \mathbf{q}$ for any $\mathbf{q}$, and therefore $\mathbf{I}\mathbf{I}' = \mathbf{I}'$. It then follows that $\mathbf{I}' = \mathbf{I}$, since multiplication has been shown to be commutative.

Theorem 5. *There is a number $\mathbf{J}$ such that $\mathbf{J}^2 = -\mathbf{I}$. The only other number satisfying this relation is $-\mathbf{J}$.*

In theorem I choose $\boldsymbol{\alpha}$ to be $\mathbf{I}$, and $\boldsymbol{\beta}$ to be a complex number $\neq n\mathbf{I}$. Then equations (1), (2), (3) become $\mathbf{I}^2 = \mathbf{I}$, $\boldsymbol{\beta}^2 = b_1\mathbf{I} + b_2\boldsymbol{\beta}$, $\mathbf{I}\boldsymbol{\beta} = \boldsymbol{\beta}$. Thus $\boldsymbol{\beta}^2 - b_2\boldsymbol{\beta} = b_1\mathbf{I}$, and therefore $(\boldsymbol{\beta} - \frac{1}{2}b_2\mathbf{I})^2 = (b_1 + \frac{1}{4}b_2^2)\mathbf{I}$. Now b_1, b_2 cannot both be zero, and from (4) $b_1 + \frac{1}{4}b_2^2 < 0$. (This is also obvious, since $b_1 + \frac{1}{4}b_2^2 > 0$ would imply $\boldsymbol{\beta} = n\mathbf{I}$ with n real.) We may therefore write $\mathbf{J}^2 = -\mathbf{I}$, where

$$\mathbf{J} = (\boldsymbol{\beta} - \tfrac{1}{2}b_2\mathbf{I})/(-b_1 - \tfrac{1}{4}b_2^2)^{\frac{1}{2}}.$$

Now if there is a $\mathbf{J}' \neq \mathbf{J}$ and such that $\mathbf{J}'^2 = -\mathbf{I}$, put $\mathbf{J}' = r\mathbf{I} + s\mathbf{J}$, where r, s are real numbers. Then $\mathbf{J}'^2 = (r^2 - s^2)\mathbf{I} + 2rs\mathbf{J} = -\mathbf{I}$, and therefore, using theorem 1, $r^2 - s^2 = -1$, $rs = 0$; this implies $r = 0$, $s = \pm 1$, since r and s are real.

Any complex number may now be expressed as $\mathbf{z} = x\mathbf{I} + y\mathbf{J}$, or $\mathbf{z} = (x, y)$

where the complex numbers (1, 0) and (0, 1) are now **I** and **J** respectively. The product becomes $\mathbf{zz}' = (xx' - yy')\mathbf{I} + (xy' + x'y)\mathbf{J}$, *i.e.*

$$(x, y)(x', y') = (xx' - yy', xy' + x'y),$$

giving the required law of multiplication. As in customary treatments, it may now be noted that **I** plays the role of ordinary unity. Any complex number may be written $z = x + iy$, and the ordinary laws of algebra may be applied with i^2 put equal to -1. R. A. N.

1610. Similarly outrageous seem to any calm considering the fictions of mathematics. This fact, indeed, was recently pointed out to me by my small son, in whom his governess was endeavoring to implant that two and two make four. But the child stayed sceptical. He was reservedly polite about a rational, "Suppose you had two apples, and I gave you two more apples, how many apples would you have then?" He conceded with readiness, not unflavored with resignation to the obtuseness of grown-up persons, that in such circumstances he would have four apples, but could not eat that many without being real sick. Yet that two and two, in consequence, make four, he excluded as a logical inference: and he depreciated that inference by stating it did not mean anything.

He was, of course, in this contention entirely, and most lucidly, correct.

For that "two and two make four" becomes, the very instant that you play this familiar axiom the childish trick of thinking about it, at best an unprovable hypothesis. That two apples and two more apples compose four apples is, as my son admitted, plain enough. Or, you may change your unit to a penny, a match, a pencil, or to a bungalow, and still produce convincing evidence to prove your arithmetic.

But the mathematician requests us to consider an abstract "two", to believe in two apples with the pomaceousness removed: his incorporal and incorporeal "two" has never existed and never can exist. His "two" is not merely a fiction: it is an inconceivable fiction which the human mind can no more, really, imagine than it can imagine his "four". You need only for one moment attempt to form some rational and clear-cut idea of this "two" to perceive that the governess in fact was (with all respect to her) talking about incredible fictions, just as my son affirmed. And when the mathematician goes on from "two" and "four" into the higher branches of his romance weaving, and postulates as yet other realities his "lines" that have length but no breadth or thickness, or his "points" that have not even length, you then face the choice between fleeing from his self-evident lunacy and accepting his insane but very useful fictions.—J. B. Cabell, "A note on alcoves" (*New Republic*, April 1922), incorporated as Chap. I of *Straws and Prayer-Books*, 1924. [Per Prof. E. H. Neville.]

1611. May the PRINCE of GEOMETRICIANS, the OMNIPOTENT CREATOR of the Universe, long preserve YOUR GRACE, and may you long enjoy that Affluence, which, like the SUN it self, affords Warmth and Life to all about you.—Henry Hill, in the Dedication of his edition (1726) of *The Six First, Together with the Eleventh and Twelfth Books, of Euclid's Elements*, "To the Most High Puissant and Noble Prince CHARLES, Duke of Somerset, . . . Chancellour of the University of Cambridge, . . . &c.", who, according to the *D.N.B.*, owed to his marriage to Elizabeth Percy "all his wealth and at least half of his importance". [Per Prof. E. H. Neville.]

1612. The new device doesn't do away with the problem, but it does cut down resonance. It is said to be 300% better than the usual shock absorber in this respect.—*Business Week*, Dec. 13, 1947. [Per Dr. W. Edwards Deming.]

NON-CERTIFICATE MATHEMATICS.*

Mrs. E. M. Williams (City of Leicester Training College) : The new examination regulations for the Certificate make this topic of greater importance than hitherto, for a large number of quite able pupils will not be entered for Mathematics when the age limit is imposed and the Pass standard becomes equivalent to the present Credit standard. For such candidates a briefer and more simple version of the School Certificate course will not do, for this syllabus has no completeness and appears to give no feeling of satisfaction. Nor does it have any obvious usefulness, though occasional " applications " give some encouragement and pleasure to the practically-minded. It is true that the algebra and geometry of School Certificate give their own particular kind of pleasure to those able to appreciate them, but they are of doubtful educative value to those who cannot do so. A course which is not designed for examination and yet is considered worthy of a place in the curriculum must have some educational justification. To know why it is included may give some clue to its desirable content.

Various reasons are given for the inclusion of Mathematics. Some people stress its use in practical affairs, either in the home or in the workshop. It is helpful in such hobbies as wireless, gardening, or running a motor-car. It has a function in civic life, in town-planning, problems of food supply, and population. Planning to meet such needs must produce diffuse and varied courses. Other people emphasise the pleasure of possessing some insight into a subject which commands so general a respect and is known to be so powerful an instrument. This suggests that we must teach techniques and topics which are commonly talked about, such as quadratic equations and Pythagoras. But to those who hold that the function of Mathematics is to enlarge the mental vision and give a deeper understanding of the relationships that exist between things and between events, it is mathematical ideas and discoveries which seem important. There are also those who think that Mathematics is a way of thinking, with its own language, which adds to the power of the mind. For them the course must include some reasoned argument and a knowledge of symbols. From these different ideas about the purpose of Mathematics in schools it appears that no single course is likely to suit all children or to please all teachers. We need not then waste time in proposing a syllabus for universal adoption. But we may usefully ask whether there are any general principles governing the framing of a course for non-Certificate classes, whether in Grammar Schools or in Secondary Modern Schools.

We will assume that Mathematics is the science of relationships, and that it has its own language in which to express them, both in particular cases and in general forms. Relationships may connect things or they may connect abstractions, such as numbers. For young children, and for many older ones, connections between things will be the most easily understood. Purely abstract relations come later. So our starting-point must be with things. Experience of many and varied relationships between things must be the groundwork. Practical work, the observation and recording of free experimentation as well as directed activities, must give the material for establishing and expressing relations of an interesting kind. This implies watching and noticing as well as doing, for Mathematics can be seen in the structure of the world around. Yet doing has a particular importance for children whose powers of abstraction and perceiving abstract ideas are limited. Mr. Durell has concentrated on this doing aspect in his book on *Modern School Geometry*, with its interesting scheme of basing all the work on the use of geometrical instruments, the ruler for lines, the protractor for angles, the set

* Discussion at the Annual Meeting, Birmingham, 1949.

square for parallels, and the compasses for circles. This suggestion will certainly prove valuable for certain parts of the course. Yet concentrating the attention on tools has its dangers. One criticism of Certificate Mathematics in the past has been that it emphasised unduly the tools and techniques of the subject to the exclusion of the truths they reveal, *e.g.* the solution of equations, logarithms, geometrical constructions. It is the purpose of a tool which is its most interesting feature ; our interest in acquiring skill with it is self-extinguishing. We must ask whether the tool is worth using and whether we want in fact to use it. The more limited the amount of Mathematics attempted, the more necessary it is that purpose and value shall be found in what is included ; for these pupils will not reach the mathematician's satisfaction in pure form.

The purposes which will appeal to Non-Certificate candidates may be of several kinds. Mathematics may be the tool which enables them to do useful practical things in, for example, woodwork, furnishing or navigation. It may provide a useful way of summarising experience in the formulae of algebra, mensuration or mechanics. It may be illuminating, throwing light on the peculiar way in which things turn out ; this is especially important for the many who will never require any but the simplest of mathematical techniques, but who will need to understand a great many mathematical principles and to have some insight into the " mystery " of the mathematician's craft. And there will be some who need a foundation of elementary mathematics because they have some modest skill in it and show signs of needing it in their future jobs.

There are two questions to consider. First, is there a core of mathematics which should be included in every Non-Certificate course? Secondly, can these courses be seen as unified courses, dealing with a single subject without seam or division? It is admitted that every teacher should plan his own syllabus to suit his particular class ; yet I believe that there are a number of mathematical principles and ideas with which boys and girls should be familiar before they leave school. Not all should learn them in the same way or in the same setting, or in the same order, for they should be learnt in connection with the children's interest and experience, which will be different for every group. Among the ideas which we should be able to take for granted I would include the following :

(1) That there is a structure in our number system and that patterns can be seen in numbers, such as odds and evens, squares and multiples, powers and logarithms, and such series of numbers as turn up in tables of interest and records of growth. Such patterns are of more service than skill in computations that will never be used.

(2) That we can generalise about numbers ; for example, we can see that square numbers, 1, 4, 9, 16, 25, ... increase by 3, 5, 7, 9, ... , and can be expressed as $2^2 + 2 \cdot 2 + 1$, $3^2 + 2 \cdot 3 + 1$, $4^2 + 2 \cdot 4 + 1$, and finally, $n^2 + 2n + 1$.

(3) That we can measure nearly all kinds of things, length, time, slope, etc. This will bring out the valuable ideas of ratio and approximation.

(4) That we can observe the process of change and see how one change depends on another ; this can be expressed as a formula or shown in a graph, and the idea of function will develop from seeing area as dependent on length and distance travelled as a function of time. The rate of change is not too difficult for these pupils if it is studied simply in relation to motion and measured on a graph.

(5) That there are relations other than those of numerical type, such as those seen in shapes and position ; horizontal, vertical, parallel are examples.

(6) That choice and chance are not entirely erratic, and that there is something to be learnt about such popular topics as football pools. Indeed, some Mathe-

matics can be learnt from the pools, as the young man in my train had discovered when he said to his girl friend, " Do you know what I have found out? It is as difficult to get all the results wrong as it is to get them all right! " But he had forgotten that there are draws as well as wins and losses. But there is a serious value in the study of the figures for accidents and epidemics, and in the realisation that they lead to prediction and control.

These are the ideas which the teacher will have in his own mind as worth while for the children to study. Now for our second question. Can they be conveyed to the children as a unified course with arithmetic, algebra, geometry, trigonometry and mechanics all welded firmly together? I believe that the answer is Yes, provided that the teacher himself knows the principles he wishes to teach and can see them for himself revealed in the things the children observe, or handle, or make. I should like to quote two practical examples of the way in which it can be done. Suppose that a class of 11-year-old children is interested in making a model of their Parish Church or in constructing a rabbit hutch. We shall find that they will be led to study:

Measurements and calculations.
Scale, ratio, fractions, approximations.
Shapes—rectangle, square, triangle—the right angle.
Position: fixing the site on the map or on the ground. N.-S. and E.-W.; horizontal, vertical; erecting a vertical post; sloping roof, angle, sine bar.

The church model may lead to finding inaccessible heights by angle measurements and drawing. The hutch may demand the estimating of quantities and costs.

A third-year class may develop an interest in water-supply, its storage and delivery. This may lead to:

Numerical work on statistics obtained from the Water Board, and the costs of supply.
Average consumption, the graph of consumption over a period and prediction of future needs.
Storage; tanks, reservoirs, volumes, conversion of cubic feet to gallons; making pipes, area and volume of cylinders.
Shape of a spout of water; comparison with the squares curve, already met in work on numbers and areas.

Many such topics will suggest themselves, and will give opportunities for the recording of experimental data, the drawing of diagrams and graphs, the building up of tables of values, and the practice of necessary techniques. In general, the aim will not be the mastery of techniques, such as finding the volumes of numerous tanks in a variety of units, but the understanding of such relations as that between a volume and the dimensions of the container, and the effect of doubling one or more of the measurements.

Understanding comes to some people more easily by a practical or a visual experience than by a numerical statement. To the non-mathematical the visual is often the more instructive. We must try, therefore, to devise non-Certificate Courses which will contain many fewer processes, much less working of " sums ", but which will lead from experiment, drawing and construction to the true joy of the mathematician, the perception of abstract truth.

Mr. R. C. Lyness, H.M.I., began by saying that he was giving his own personal views. There was not as far as he knew any official view on the right content of the mathematical curriculum. He had listened to Mrs. Williams carefully in the hope of finding something to disagree with her over, but in her excellent account of the problem he could only find one trivial thing to criticise. He considered that the definition of Mathematics as the

science of relationships was too general. *Quantitative* relationships perhaps, but relationships such as that of father to son or truth to beauty were not usually considered the subject-matter of mathematics.

In teaching mathematics the teacher in the Modern School had an advantage in that he often taught some other subject too, and valuable mathematics could arise naturally out of other subjects such as Geography, Science, Handicraft, Gardening and ways of representing quantitative information. He suggested that one line of approach was to consider mathematical topics, and gave examples such as functionality, similarity, numerical accuracy, the nature of proof. Some of these things sounded more difficult in the abstract than they were in practice. Every child should get some idea of what deductive proof meant. One assumed certain things and argued from them to new conclusions, which depended for their validity on the truth of the initial assumptions and the rules of argument. Children should be given as much opportunity as possible for discovering things for themselves, *e.g.* they might be left to find out the sum of the first so many odd numbers and discover the general result; or they might be asked to estimate how much water would be wasted if an actual tap dripping in the laboratory were left to drip all night. The important things were to *use* mathematics and to get insight into ideas about and relations between quantities. He hoped that in the ensuing discussion time would not be spent on problems of organisation, but on the more interesting problem of what sort of mathematics should be done.

Mr. B. J. F. Dorrington (Camden Training College, N.W. 1) endorsed all that Mrs. Williams had said, and supported her most strongly. The Emergency Training Colleges were making a valuable and effective contribution along the lines laid down by the speaker, but progress would of necessity be slow. So often the mastery of technique loomed so large that there was little attempt to expand the mathematics course along the lines suggested by Mrs. Williams. How was this change to be effected in the whole of the profession?

The particular topic such as the rabbit hutch *was* very difficult to deal with, but the idea could be pursued in a narrower sense very successfully, provided it was fitted very carefully into the scheme of work and not added on as an extra.

He suggested a more widespread consideration of the mathematics involved in Earth Measurements and Elementary Astronomy. Latitude and longitude provided examples in *real* angles, and there were many *real* mathematical relationships that could be considered under these topics.

Mr. M. Pitt (County Modern School, Stoke-on-Trent) thought that the Emergency Trained Students were definitely going to reinforce the teaching profession, but that they sometimes suffered from over-enthusiasm and wanted to do more than the child could manage. The established teachers tried to help these newcomers and did not damp their enthusiasm.

Mr. Dorrington replied that, generally speaking, the Emergency Trained Students had received every help and encouragement in the schools, but that the problem of getting these new ideas generally accepted still remained.

Mr. J. W. Withrington, H.M.I., "speaking for himself", was fond of paradoxes. The scheme of work had taken a very secondary place in the past, and this had not mattered so very much because most teachers followed a text-book and so had a logical order worked out for them. If the more modern and exciting approach was practised, the scheme of work now became of paramount importance in ensuring that the simple ground was being covered and that no large holes were being left in the syllabus.

Mr. I. R. Vesselo (Alsager Training College) said there is no real answer to the question, "What is Maths?" Mrs. Williams thought it should be enjoy-

able, and Mr. Lyness suggested an approach through other subjects. Had we to give the children ideas or a technique, to learn the tools or how to use them? It does not matter to a child whether he can get 10 sums right out of 10 or not in making him a useful citizen, and a boy who cannot do tables should not be debarred from other mathematics. In the Modern Secondary School learning technique only was not enough, and there was much else that was of value to every child. The position was similar to that of music-teaching in the old days and the changed approach to-day.

Miss A. M. Trout (Southampton) did not agree that it does not matter whether a woman gets the correct change when shopping, or that a man should not know the exact amount expended on pools. Technique and accuracy was essential for every child.

Mr. L. G. Edgcombe (Plymouth Technical Secondary School) followed a similar line of thought, and felt the children must know how to handle figures in their calculations. He was sorry to see the question of technique so reduced, and children who went on to further study in Evening Classes or Day Continuation Schools needed it. The ability to use reference tables was also important. He was pleased to see the emotional angle of mathematics stressed, and this fitted in with the recent pronouncement of the Prime Minister that all teachers should know more of the world in which the children would eventually live and work. Syllabuses will be dictated not by general requirements, but by the area in which one works.

Mr. R. H. Cripwell (Didsbury Training College) thought that more graphical work was needed. There was too much stress on the computational side of graphs and too little on the informational. He pleaded for the ability to interpret a graph rather than to attain skill in drawing one.

Mr. H. V. Lowry (Woolwich Polytechnic) thought it would probably be twenty years before there were enough Technical Schools to cope with the numbers requiring this sort of education. In the meantime every child should know how to use Mathematical Tables, and this was something that could well be included in the curriculum.

Mr. T. Lloyd Phillips (Tregallan County School) thought our approach tended to be idealistic, and asked whether the less intelligent pupils could do formal geometry, whereas they could do constructional geometry. For those who could reason in geometry it was worth while, because this would help them in solving other problems.

Mr. T. R. Theakston (Coventry Training College) disagreed, and said we do not teach mathematics to enable a child to reason in other subjects. It does not follow that there is any transfer of reasoning power. Mathematics is taught firstly for a bread and butter reason—a small one—and secondly, as one more part of Bacon's "full life". The children should be taught mathematics so that they like it, and more history and number work should be included.

Mr. H. F. Marks (King's School, Grantham) felt we were dealing too much with the top stream of the modern school. We should find out what the child is going to do and suit the mathematics accordingly. There was a need for mathematics of citizenship, the ability to understand mathematics in newspapers, the formation of companies, stocks and shares, etc., without necessarily being able to do the actual problems. There was something in human nature which made a demand for mathematics, and the history of the subject should be studied so that the growth of our civilisation could be understood.

Mrs. Milne (Day Continuation School) said there was a noticeable improvement in the standard of the children who went to the Day Continuation School after an extra year in the Modern Secondary School.

The extra year was definitely worth while in her experience. She had compromised between the formal development and the extreme topic method, and had covered successfully much of the ground advocated by Mrs. Williams. Geometry was approached from the angle, " How to find my way about " ; and money problems were essential, both to the individual and from a social aspect.

Mr. Alexander (Middleport School, Stoke-on-Trent) pointed out that the pupils in Modern Secondary Schools had been creamed, and we should deal with the sort of mathematics they would need in everyday life. The surroundings in which the school was situated would dictate the content of syllabus and the method of approach.

Mr. F. J. Burfoot (Wolverhampton Secondary Modern School) said that a large number of secondary modern school teachers were present, and they were grateful for the experience, for the vision, and the opportunity to pick up ideas. Much could be got out of mathematics : beauty, history, and utility. All these aspects were needed. We should all do well to write down twenty reasons for teaching mathematics and read it monthly—hang it on the bedroom wall like a text! We needed a great vision, and some of it would then be passed on to the children.

Mr. J. T. Combridge (King's College, London) thought there was an impression that a cast-iron syllabus for the Modern Secondary School was needed in the next twelve months or so. He did not agree. There were no text-books as yet, and he hoped there would not be any for a long time. We never get the same type and child every year, and there was a tendency to follow the *A*'s instead of the average. Modesty must be overcome, and ideas, useful dodges, etc., should be sent in to the Editor of the *Gazette*. It was cheaper and better to buy the *Gazette* than a host of new text-books in order to keep up to date.

Mr. A. Blackwell (Ecclesfield Grammar School) endorsed all that Mrs. Williams had said, and made a plea for the introduction of the principles of statistics and an elementary treatment of it. This would render a great service to the community.

Mrs. Williams in reply said that, arising out of the first question and other remarks, the problem seemed to be how to teach interesting topics to pupils lacking technique. She felt that teachers were often afraid to embark on new methods. Students should be encouraged to go along these lines, but go slowly, and the teacher who knows his class better than the student can be of enormous help here.

She felt she had been misunderstood over the matter of techniques. Children should know the number bonds absolutely, and the first job of the Secondary Modern School was to see that they were known. Techniques served two purposes :

(1) To give the child a useful tool for later use. It was not the job of the Secondary Modern School to give special techniques for a particular job—that was for the Technical School or County College to do. We must have a wider view, although local industries were likely to determine the bias of the syllabus.

(2) By practising a technique one got to know the idea lying behind it. This was of value in itself to the child as a person, and was fundamentally important in everyday life.

The course for the *C* stream would differ according to the capacity of the child, and the interest should not be limited to number only ; relationships should be stressed. The introduction to reality for the *C* child was not made through number, but through the things which they did with their hands.

The Chairman, Mr. W. J. Langford (Battersea Grammar School) summed up by saying there was a minor revolution in the Grammar Schools, and the addresses which had been heard would help in tackling this revolution. The Association's Teaching Sub-Committee, of which Mrs. Williams was Chairman, was producing a report, but her enthusiasm, as shown to-day, would not let the document be published until it was complete. A digest would soon be available. He then thanked the speakers for very inspiring addresses in opening the discussion, which had proved long and useful.

1613. I think of Miss Greub and her sums on the blackboard, which never came out wrong. Stupid figures ; how I loathed them, and loathe them still. What a mess mathematics make of man, damming his generous currents, frowning on joyous fallibility, pursing the dry lip at admirable error. They say that Einstein employs underlings to handle his figure-reckonings. That must be why he can still enjoy human intercourse and can smile without a dry creaking sound.—R. C. Robertson-Glasgow, *46 Not Out*, p. 7. [Per Mr. C. E. Kemp.]

1614. It may have been observed that the political, like any other, pendulum swings evenly between extreme points of amplitude. If the pendulum has swung extreme left to Socialism it may legitimately be inferred it will swing back extreme right to Conservatism. The tempo of politics will have to die down considerably before the amplitude will be co-extensive with Liberalism, from tepid left to lukewarm right.—The Hon. Arnold Keppel, in a letter to the *Daily Telegraph*, Dec. 11, 1947.

1615. *Philosopher :* I have been thinking of the procopo.

Fool : So have I. What is it?

Philosopher : The procopo is a small Brazilian animal chiefly remarkable for singularity of diet. A procopo eats nothing but other procopos. As these are not easily obtainable the annual mortality from starvation is very great. As a result there are fewer mouths to feed and by consequence the race is rapidly multiplying.

Fool : From whom had you this?

Philosopher : A professor of political economy.

Fool : Let us rise and uncover.—Ambrose Bierce. Quoted in the *Sunday Times*. [Per Professor Milne-Thomson.]

1616. A suitable stone was brought in from outside and Marshall dropped it down the well and began to count, starting from zero as the stone left his hand. Finally there was a deep resonant " plonk ".

" That's a long way down," he said. " Nearly three seconds, if my idea of seconds was correct."

" How deep? " asked Denise.

" Sixteen feet the first second, then thirty-two for every succeeding second, plus initial velocity. I make that close on a hundred and sixty feet."

" How clever! "

" Yes, but all the credit goes to Isaac Newton."—George Goodchild, *Brave Interlude*. [Per Mr. G. A. Garreau.]

MATHEMATICAL TEACHING IN OTHER LANDS.

At the Annual Meeting of the Association at Birmingham in April 1949, a morning was devoted to a discussion opened by Mr. R. G. Keats and Mr. B. T. Gilroy, of Australia, and by Dr. H. C. Christofferson, of the U.S.A., in which the visitors described methods and ideals of mathematical teaching in their respective countries.

TEACHING OF MATHEMATICS IN AUSTRALIA.

Survey of Educational Systems.

By R. G. Keats and B. T. Gilroy.

Primary and Secondary education in Australia is now and has always been exclusively the business of the various States. There exists no body for co-ordinating the various state departments, since there is no Federal body to which these departments are subordinate. Hence the detailed organisation, syllabi and external examination systems vary from State to State, although in all States at the end of secondary school the average ages and standards in most subjects are roughly the same. So that despite the State-basis of the educational organisation, the divergences in primary and secondary education between the States are really less than one might expect. This is no doubt due to the free interchange of ideas within Australia and our comparative isolation from other countries.

Education is compulsory for all Australian children between the ages of 6 and 15 years. In addition the education provided by the States is "free and secular" ("free" in the accepted sense in which all social services are free). The education within the State is the special responsibility of a Minister of Education. A permanent public servant, the Director of Education, actually is responsible for the detailed administration of the department, and helps and advises the Minister in the interpretation of his Government's policies.

The States' control of education is highly centralised and extremely rigid. The Director is kept informed by a corps of inspectors, who advise teachers on educational developments and techniques, as well as making reports on the state of buildings and effectiveness of teaching. These reports are systematically kept for reference.

The importance of this painstaking procedure can be appreciated when one remembers the vast distances between centres of population in Australia. In the country there are a large number of schools which are small one-teacher units, providing courses in all the subjects for all the children for all of the compulsory years. Correspondence courses are conducted by specially trained teachers for those children too remote even for these rural schools.

Secondary schools are provided wherever there are sufficient numbers of pupils to warrant it. In country areas these are associated with primary schools, in towns they function also as technical and home science schools; but in the cities and large towns one finds the high schools proper, whose specific task is preparing pupils for the University or other forms of higher education.

The whole system of state education in Australia is duplicated by the Catholic Church—uniformity in secular subjects is assured by the certification of the teachers and the same corps of inspectors. All pupils take the same public examinations.

Other denominations also provide schools, and in general the denominational schools, which cater for boarders, have considerable prestige, comparable with the public schools in Great Britain.

Technical education is available for apprentices and others wishing to do specialised technical work. The technical colleges and schools of mines provide both full and part-time courses. In some cases the diploma courses provide the only training available in certain fields and hence overlap with the universities. The universities, unlike the technical colleges, are administered by self-governing councils.

Development and Reform.

Recent years have been notable for the marked increase of the activities of the Federal Government in the educational sphere. In 1945 the Federal Government founded its own Office of Education to provide officers for research into teaching methods, and also to act in an advisory capacity to help the Federal Government handle its enormous commitments in the field of post-war rehabilitation. Also during the war the Commonwealth established a Universities Commission to give direct financial aid to the students in order to stimulate the then inadequate supply of trained professional men. This arrangement has been continued into post-war years.

Education reforms which are at present proposed in Australia will require some 6,000 new teachers—and we think it is true to say that the number of teachers in Australia has never been adequate, and the qualification requirements which obtain now are higher than they have ever been.

Secondary school teachers are required to have a degree, and teachers in the technical colleges require a first-class degree, although their other qualifications, *i.e.* experience, knowledge of method, etc., could hardly be more varied.

Mathematics in Secondary Schools.

The reforms we have cited above cover the field of mathematics, and indeed a very serious effort is being currently made (1) to provide the average child with his mathematics in such a way as to give him really useful equipment for his everyday life, and (2) to teach mathematics as a science with intrinsic potentialities and an importance *per se*. This is being undertaken by providing divergent courses for these two classes of students. For the first group is provided a single course termed " General Mathematics ", where arithmetic is of prime importance, algebra is regarded as a convenient shorthand, and both geometry and trigonometry are of a descriptive and practical character. No serious effort is made to teach the pupil to do the mathematical act as such.

On the other hand, the second group, who are visualised as rather more specialist—those in fact who will pursue the subject beyond the secondary school stage—are taught Mathematics proper, termed Mathematics I and Mathematics II. Now arithmetic is regarded more as a medium for familiarising our young scientist with the object of his science. There is no artificial barrier between arithmetic and algebra, and he is soon brought to the idea of function, and so he proceeds to unravel the properties of the functions which constitute his first exercises in what is simply mathematics.

The first and second years courses in the secondary schools have been modified in an endeavour to make them simpler and more palatable for those students who will not be pursuing the study of Mathematics any further. This tendency is again evident at the Matriculation standard in some States where a higher standard is required of the student who will be entering Science, or Engineering or some other faculty where Mathematics is essential, than that required of the student who will be leaving school or entering some faculty where Mathematics is less essential.

The public examination system is in evidence in all States, and usually

consists of an Intermediate Examination after three years, a Leaving Examination after another one or two years, and Leaving Honours Examination. The Leaving Examination is often the standard required for Matriculation purposes, and is of rather higher standard than that required for the English School Certificate. Conversely, the Leaving Honours standard is probably not as high as the English Higher School Certificate.

In some States the Internal Examination system is firmly established, and in these States the student may take only one external examination. The teacher sets these internal examinations, marks the papers and makes recommendations as to pass or failure. The work done by the student during the year may well be considered before such a recommendation is made.

About 60% of those students reaching Intermediate level will continue with the leaving course, and the remaining 40% will probably seek employment. Apparently the External Examination carries more weight with employers, and even in those States where the Internal Examination system is well established some form of External Examination may be set to meet the needs of such students.

The amount of time spent on Geometry, especially Solid Geometry, is being steadily reduced to allow of time being given to Calculus, which is so important to engineers and physicists. The theory of functions of a complex variable must be introduced at the Second Year University Standard, to enable the engineer and physicist to learn a little of Contour Integration and Conformal Representation. This can only be done by introducing the study of Calculus at an earlier stage. Thus in most States Calculus is introduced at the Leaving Standard, and unfortunately the time devoted to Geometry is curtailed.

The emphasis is placed at all times on the comprehension of fundamental principles rather than on the attainment of manipulative skill, and teachers are urged to consult the reports of the Mathematical Association for the principles and teaching methods. These reports are the main source of information as far as the teacher is concerned.

In spite of the vast areas to be covered by the Education system, it is possible for almost all students wishing to do so to attend a Secondary School to Matriculation Standard. Leaving Honours, however, is in general confined to the cities, while most States have only one or two universities. It is not until University standard is reached that the country student finds himself at a real disadvantage, even though Government assistance is available to most students who have matriculated.

University Ordinary Degree standards are comparable with those of England, but the number of students attempting an Honours Degree is probably much fewer in comparison. Some increase in this number has occurred in recent years, and with the present expansion of secondary industries and research facilities in all States, the opportunities for the mathematician should increase considerably.

R. G. KEATS and B. T. GILROY.

BASIC CONSIDERATIONS IN THE TEACHING OF MATHEMATICS IN THE HIGH SCHOOLS OF THE U.S.A.

BY H. C. CHRISTOFFERSON.

GREETINGS from your brother organisation in America, the National Council of Teachers of Mathematics, I bring to you. This I do as a former president of that organisation and at the request of Dr. Reeve. I also bring you greetings from Dr. Reeve, and assure you that the honour which you bestowed upon him by election to honorary membership in the British Mathematical

Association will be a source of much joy to him. I was a member of his first " Methods " class and first group of student teachers. I have been a devoted disciple, and can assure you that Dr. Reeve was a " methods " teacher who " knew his mathematics ". The fine appreciations which you have given him as editor, teacher, educational leader and person are well deserved and highly fitting.

You little realise what a joy it is to speak in English to an English-speaking audience, even though the English I use is American English and may not always have the same meaning for the words used. Before discussing Mathematics teaching in America, I must confess that I was not wholly unselfish in coming to this conference. I came partly, perhaps mostly, to see England again and to meet some of those men whose books and articles I have often read and concerning whom Dr. Reeve often spoke. Siddons, Durell, Robson and others are now no longer mere names on the cover of a book. Like my Australian friends, I too feel as though I were a student addressing my teachers. I am delighted with the fine sense of humour which you have shown. Britishers are not supposed to be witty. I have had several delightful corrections concerning impressions of England. In fact, I can now appreciate a story which has been circulating. A G.I. was injured and brought to an English hospital. A sister came into his room and seemed very quiet. The G.I. said : " Sister, what are you doing? "

Sister : " I am praying that you may get to Heaven."

G.I. : " Sister! I don't want to go to Heaven, I'd rather stay in England."

It is with fear and trembling that I generalise about teaching mathematics in America. It would be perhaps as safe to generalise about England from the experience which I have had. Having seen the movie, " Mrs. Miniver ", I should know that most Britishers have private yachts. Having visited King Edward VI Grammar School, and the University Hospital in Birmingham, both built in 1938, I now know what the grammar schools and hospitals are like. They are elegantly, efficiently equipped, and modern in all respects. In the boys' school, the teachers stay put and the class moves about ; while in the girls' school the classes stay put and the teachers move from class to class.

There are several aspects of American education that I would like to discuss, such as " Decimalization of our English Units of Measure ", " Mathematics which has become obsolete in the last forty years ", " Two-Track System ", " General Mathematics ", " The Core Theorems of Geometry ", or " Geometry, a Way of Thinking ". From the discussions of this group, these subjects are vital here as well as in America. However, in order to supply you a picture that would be the most helpful for you to appreciate our problems and progress, I have chosen to emphasise certain basic considerations and to illustrate their meaning.

American teachers of mathematics owe an unpayable debt of gratitude to great English teachers of mathematics, especially to those who have written text-books during the last several centuries. I shall not attempt here to mention their names. To deal adequately with their contributions to American education would require a lengthy treatise. Dr. Eric Bell in his recent book, *Men of Mathematics*, has done this in part. However, we owe such a conspicuous debt to pioneers like John Perry, who began the battle for a functional mathematics, that I must mention his name. T. Percy Nunn, through his book on the *Teaching of Algebra, including Trigonometry*, has provided a philosophy of mathematics and a basic teaching technique for which all American teachers are extremely grateful. His book is the Bible from which teachers in Teachers Colleges often choose their texts. Dr. Hamley's treatment of the function concept in the teaching of Secondary School Mathematics

has served as a trail blazer and stimulus for many followers who are trying to make mathematics more than a bag of tricks, more than an accumulation of important yet often meaningless techniques.

In order that you may understand what American high school teachers of mathematics are doing, it is necessary that you appreciate the basic problems facing these teachers in the selection of materials and techniques. Two basic ideas or facts form this background, and may perhaps be called educational postulates for teachers of mathematics in secondary schools of the U.S.A.

The first deals with the enormous increase, during the last forty years, in the number of children and in the percentage of children attending high school. In 1900, a little over half a million boys and girls were attending our high schools; 1910—about 1,000,000; 1920—2,000,000; 1930—4,000,000; 1940—6,000,000. In my State of Ohio 65% of those entering the fifth year of elementary school complete the twelfth year of education by graduation from high school. Of these high school graduates only about 15% go on to college or university. Therefore, the high schools must adjust their offerings so as to serve best the needs of this large number of young people, 85% of whom do not go on to college.

One might say that the standard has been lowered. This is erroneous. It would be possibly more correct to say that the standards for the selection of subject-matter and teaching techniques have been raised. Let me illustrate by means of a story from Germany.

A teacher in a gymnasium, Dr. Lang, was watching two boys in a vacant lot trying to fly kites. For convenience we shall call them Hans and Klaus. Hans got his kite up very easily and it held its position beautifully. In spite of all the oral help Hans could give him, Klaus just could not fly his kite. Finally Hans said: "Here, Klaus, hold my kite." Hans picked up the other kite, adjusted the strings a bit, changed the tail, and behold the kite took off and flew as well as his own. Dr. Lang became interested in the boys, and went over to see them. Klaus, who could not fly his own kite, was one of the "Begabten", the gifted children, from a gymnasium. Hans was from a Volksschule, where the other children finish their education. Supposedly he was "unbegabt", not gifted. Klaus could recite his Latin declensions and conjugations and his mathematical theorems and formulas, but Hans could fly a kite. Dr. Lang was broad-minded enough to ask: "Wer war das begabte Kind fuer das Deutschland von heute?" Which was the gifted child for the Germany of to-day? Would the level for German education be lowered or raised by including boys like Hans in the Gymnasium? We will avoid that question by saying that the material would be different, perhaps neither lower nor higher.

This is the problem faced by American teachers of mathematics. We, too, have boys like Klaus, bright little boys who do what teacher tells them to do, who can recite their lessons perfectly, who have keen academic minds. We have also a lot of little boys like Hans, who can fly kites, who are interested in ideas which serve a purpose, who like to understand rather than merely to memorise. Out of 6,800,000 boys and girls, most of ours are like Hans, a bit practical-minded.

The second major educational postulate deals with a philosophy or psychology of education. We do believe in the disciplinary value of education, but we do not accept the conclusion that the disciplinary value of subject-matter is directly proportional to the age of the material, nor that any traditional material has a monopoly on discipline. It is possible that to design and to make a piece of furniture may have more real mental discipline than to learn that "Gaulia est divisa in partes tres". Certainly Latin, and even Mathematics, can be taught so as to have zero or actually negative disciplinary

values. Even blacksmithing can be so taught as to have excellent disciplinary value.

Useful, functional, important and powerful ideas, techniques, or procedures should have two values. If properly taught, (1) they will give discipline through the study of their inter-relationships and of their basic organisation, and (2) they will have practical values, as well as to add interest and zest to school life. With this very inadequate treatment of the frame of reference from which mathematics teaching in the high schools of America must be judged, my remaining time will be used for illustrating techniques and materials. However, a major principle of teaching mathematical processes, which is a much-discussed ideal, may be briefly stated as follows.

Mathematical techniques should be understood and so related to other techniques that mathematics becomes a system of consistent and related operations which are basic to quantitative thinking. A corollary principle would emphasise the concrete and practical setting for the technique; that is, whenever possible a new process is presented in a real and important problem, so that its usefulness is illustrated and its mathematical relationship to other processes is revealed. Meaningful teaching of useful materials seems an inescapably basic principle.

The following illustrations may reveal the presence of valued techniques in a setting which emphasises relational, systematic thinking, and in a problem situation which is functional as well as mathematical.

A child being taught $8+7$ is not told that the sum is 15, and then drilled on this meaningless fact until response is automatic like learning the first and last names of a friend. He is helped to find the sum by relating $7+8$ to $8+8$ or $7+7$ or $7+5$ or even $7+3$, which he may have had previously. Drill will be given on this fact after he "figures it out". He may "figure it out" many times, but eventually he will respond automatically and not always use round-about methods in adding $7+8$. Emphasis upon related facts, however, is essential for interest and systematic learning.

Teaching division by a decimal is often taught as a process wholly unrelated to previous processes. Ordinarily one has taught division by a common fraction previously. The relationship to a decimal divisor is direct and simple. The reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$, but what is the reciprocal of the decimal ·7? By simple extension of the thinking involved in dividing by $\frac{5}{8}$, division by ·7 can be done by multiplying by the reciprocal, $\frac{10}{7}$. Even though other analyses may be more lucrative for emphasising interrelationships and systematic thinking, this is one comparison and extension which should not be overlooked.

In algebra we have much to do with equations. Often, as mathematics teachers, we have tried to show the value and importance of this powerful instrument through using it to find how soon the hound can catch the hare, at exactly what tenth or eleventh of a second the hands of a clock will be together, or how long it will take Mary and John to do a piece of work together which already has been done by Mary in five hours and a second time by John in six hours. To use fine techniques in the solution of such impractical problems and to get answers which are often in a real sense incorrect is no endorsement for mathematics. Through carefully-planned division of labour, two people, because of increased efficiency, can possibly do, not 2, but 5 or 10 times as much work as one in a given time. Yet in some work two may consume as much time as one. For example: "If *one* man can climb the Eiffel Tower in *two* hours, how long will it take ten men to climb the tower?"

There is much private flying in two- and four-passenger planes, consequently a radius-of-action problem is appreciated and important. A flyer computes that he can go out on a certain cruise, with the wind, at 70 m.p.h. and return against the wind, at 50 m.p.h. If he starts with a 4-hour supply

of gas, how far out dare he fly and yet be able to return safely? One solution of 2 hours out and 2 hours back is soon discovered faulty. A second try, using the erroneous average speed, which seems 60 m.p.h., may be made. 4 hr. at 60 m.p.h. allows 240 min. or 120 mi. out and 120 mi. back. Suppose we check it. 120 mi. at 70 m.p.h. requires $\frac{120}{70}$ or 1 hr. 43 min. 120 mi. back at 50 m.p.h. requires $\frac{120}{50}$ or 2 hr. 24 min. Then 1 hr. 43 min. and 2 hr. 24 min. = 4 hr. 7 min. His safety surplus may carry him for this extra 7 min., but that is poor mathematics and gambling with death. There should be an easy, accurate way to do this. Algebra provides such an easy way.

Let the letter t = the time flying out ;

then $4 - t$ = time back.

But $70t = 50\ (4 - t)$ to return safely,

or $70t = 200 - 50t$.

Then $120t = 200$ and $t = 1\frac{2}{3}$.

Therefore the time out must be 1 hr. 40 min. and the distance out $1\frac{2}{3} \times 70$ mi. or $116\frac{2}{3}$ mi. Returning at 50 m.p.h. requires $\frac{116\frac{2}{3}}{50}$ or $2\frac{1}{3}$.

Therefore the time back is 2 hr. 20 min. As a check, now 1 hr. 40 min. + 2 hr. 20 min. = 4 hr. Dr. Hamley and Dr. Nunn would now have the student see if he can generalise this solution and get a simple formula to use. We, too, might do that, depending on the situation. In fact, we often do it by analysis of a specific case through such questions as the following.

The $1\frac{2}{3}$ came from $\frac{200}{120}$, but where do the 200 and the 120 come from?

$200 = 4 \times 50$ = total time (T) times rate back (r).

$120 = 70 + 50$ = rate out (R) + rate back (r).

Therefore $t = \frac{Tr}{R+r}$ or $\frac{r}{R+r} \cdot T$.

Check : $t = \frac{50}{120} \cdot 4$ h. $= 1\frac{2}{3}$ hr.

Here is the heart of the radius-of-action flight formula.

A concrete mix is 1–2–4, cement-sand-gravel. How much of each is required per 1000 cu. yd. of concrete? If the sand and cement would not fill in the void spaces in the gravel the problem would be very easy. But a simple water test reveals the sand 20% void and the gravel 40% void. Therefore only 80% of the sand and 60% of the gravel is solid matter. Therefore, if X represents the number of cu. ft. of cement required, $2X$ = the sand, and $4X$ = the gravel, then $X + 2X + 4X = 1000$ might be an erroneous trial. An alert Hans would quickly detect this superficial thinking. Finally,

$$X + \cdot 80(2X) + \cdot 60(4X) = 1000.$$

$$X + 1 \cdot 6X + 2 \cdot 4X = 1000.$$

$$5X = 1000.$$

$X = 200$, No. cu. yd. cement.

$2X = 400$, No. cu. yd. sand.

$4X = 800$, No. cu. yd. gravel.

It is evident that it takes 1400 cu. yd. of materials for each 1000 cu. yd. of concrete.

The beauty of this mathematics, the clarity of the thinking involved, and the educational values both in discipline and in direct usefulness of the processes learned are all enhanced by the practical setting, even though the concrete setting involves making concrete. To check this answer and to

discover a general formula are also vital and valuable educational experiences.

A hardware merchant in taking inventory has a difficult task with linoleum. He must lay the roll on the floor, unroll it and measure the yardage. There is a short, mathematical, labour-saving method of doing this job which is really fun to have children discover by means of a bit of functional thinking. The roll of linoleum represents a spiral, yet the loops are practically circles and, since they differ by a constant in diameter, their circumferences form an arithmetic series whose sum is the arithmetic mean circumference multiplied by the number of loops. The mean is half the sum of the outer and inner loops.

$$\frac{\pi d + \pi D}{2} n = L \quad \text{or} \quad L = \frac{\pi n (D+d)}{2}.$$

Now it is unlikely that there will be an integral number of loops. There will usually be a part loop (p) on the outside which can readily be measured. Furthermore, the diameters are measured in inches and the length is wanted in yards. Therefore a better formula is :

$$L \text{ in yds.} = \frac{\pi n}{2} \frac{(D'' + d'')}{36} + p \text{ yd.} = \frac{\pi n}{72} (D'' + d'') + p \text{ yd.},$$

$$L \text{ (in yd.)} = \cdot 044 n (D + d \text{ (both in inches)}) + p \text{ yd.}$$

These illustrations will perhaps suggest a trend toward mathematics that is useful as well as rigorous, that reveals functional dependence based upon mathematical principles and theory, that uses related techniques and knowledge based upon a system of thinking, and that has educational values which depend upon rigorous logical thinking in terms of used and useful data and techniques.

In the field of geometry the same two trends also appear, but there is an added conspicuous development. Text-books of the last ten to twenty years show a striking contrast to older text-books in the greater emphasis upon practical problems and in the liberal use of algebra and arithmetic, even including fractions and decimals. Less stress is laid upon a minimum list of postulates, but more upon the necessity for postulates in any system of thinking and more stress upon a minimum list of core theorems. Gear speeds and cutting speeds, bridges and highways, automobiles and airplanes seem now attractively mingled with parallels and perpendiculars, triangles and circles, theorems and postulates. They give geometry a concreteness and a usefulness, as well as essential rigour and logical sequence, that make it an interesting, useful discipline. These features are here treated very briefly in order to leave more time for " geometry as a way of thinking ". The greatest contribution to this aspect of geometry has been made by Dr. Harold Fawcett through his book, *The Nature of Proof*, through numerous published articles, many addresses, and through his classes at Ohio State University.

Dr. Fawcett may begin his geometry class by discussing a political speech, an editorial, or a sermon which his class may have heard or read. They may spend several days analyzing such material to discover the postulates upon which the thinking was based. The statements, " We hold these truths to be self-evident : that all men are created equal, that they are endowed by their Creator with certain unalienable rights, etc. ", a quotation from our Declaration of Independence, is really the system of postulates upon which our governmental thinking is based. Communism has a different system of basic principles, therefore its conclusions, even when based on correct logic, will differ from ours. Similarly, the Apostles' Creed, " I believe in God the Father the maker of heaven and earth, and in Jesus Christ his only son our Lord, who was ... etc.", is a postulate base for religion.

The rules of a game—in baseball: "Three strikes, batter out"; "four balls, take your base"; "three outs, side out", etc.—reveal the assumptions which have been accepted and with which thinking and acting in the game must be done.

Most of these systems of thinking have serious weaknesses. They are affected by prejudices and personal opinions, and are subject to emotional reactions which often interfere with logic. However, Dr. Fawcett would point out, there has been handed down to us from beginnings 4000 years ago a system of thinking that has a minimum of prejudice and emotion, a system that is objective and abstract, yet has many concrete and useful applications. This system is called Demonstrative Geometry or Demonstrative Euclidean Geometry, named from Euclid, who wrote the first text-book for the subject, Euclid's *Elements*, over 2000 years ago. Through a study of geometry a student should learn two types of ideas. First, he will learn many practical facts, ideas, principles and laws governing shape, size and position of objects. Geometry is in every object we see and use. Buildings are filled with rectangles, parallels, and perpendiculars; machines use circles and straight lines, even plants and animals utilise amazingly complex geometric principles and design. Second, an alert student will discover how conclusions are proved. He will learn the need and place for definitions. He will appreciate the nature and function of postulates or axioms, of basic assumptions or rules, of constitutions or creeds. He will understand that proof consists in showing that a certain conclusion is consistent with the assumptions and definitions which have been previously accepted.

"The simple concepts with which geometry deals give it a peculiar function. Geometry achieves its highest possibilities if, in addition to its direct and practical usefulness, it can establish a pattern of reasoning; if it can develop the power to think clearly in geometric situations, and to use the same discrimination in non-geometric situations; if it can develop the power to generalise with caution from specific cases and to realise the force and all-inclusiveness of deductive statements; if it can develop an appreciation of the place and function of definitions and postulates in the proof of any conclusion, geometric or non-geometric; if it can develop an attitude of mind which tends always to analyse situations, to understand their inter-relationships, to question hasty conclusions, to express clearly, precisely, and accurately non-geometric as well as geometric ideas.

"There seem to be certain ordinary, practical, direct values which are easy to get, and also some superior, transcendent values which are possible but not certain. These superior values depend greatly, perhaps largely, upon the way geometry is taught, and consequently are not attained by all teachers, nor by all classes, nor perhaps completely by any teacher or class. They constitute an ideal and depend upon the realisation that geometry is not a bag of tricks to be performed, not merely information to be learned, nor is it a list of rules to be memorised. It is rather a fundamental system of logic to be understood, it is an organisation of universal truth to be appreciated, it is a pattern of reasoning to be emulated."

Conclusion.

Teachers of mathematics in the U.S. are deeply indebted to European teachers for the rich background and the useful heritage to which they have fallen heir. Faced with the problem of mass education on a scale never before known, they have not hesitated to use this rich heritage, nor have they hesitated to change it in such a way as to make it meet more adequately the problem of serving the needs of the young people whom they are teaching. They are dominated with the ideal that young men and women are not empty

vessels to be filled with the social heritage of the past, but living organisms who through education can be helped to live happy and effective lives. To achieve this ideal requires the most powerful, the most widely useful, the most perfectly organised and the most widely taught subject-matter possible. Mathematics can meet these criteria if teachers are devoted, not to tradition, but to service ; not to subject-matter, but to children ; not to the memorising of meaningless formulas and techniques, but to helping the child develop into a confident, effective person through experiences which are meaningful and functional. H. C. CHRISTOFFERSON.

1617. There is no machinery within the compass of the brain of man, or within the mastery of his most skilful manipulation, that for rapidity, speed, or velocity, can approach, for a moment, to the multiplication of that frightful special-train pace by which a man goes headlong to his ruin. The road is open and pleasant at first, with leafy resting-places ; the song of birds, and the dulcimer tinkling of falling waters, making a lulling music. The *incline* is barely perceptible ; the gradient is as one in six hundred, let us say ; and after that holiday-time, when the hours pass by in mirth and laughter, and the sense of increased *momentum* is required, in order to give a fillip to what might become monotonous, the incline deflexes—the steepness increases, the movement is gathering—until at last, like a train that has ceased to answer to the brake, the acceleration grows, and a frightful, unavoidable smash-up follows ; the horrible catastrophe telling the story of the want of government, want of order, want of " brakes ", want of prudence, want of manliness utterly ; and the chronicle is over, only to be repeated by way of caution to others about to follow the same road.—Descriptive letterpress to " The Rake's Progress ", in the *Complete Works* of William Hogarth. [Per Mr. G. V. Groves.]

1618. Measure an angle of creation, make a proportionally progressive multiplication, and all infinity shall multiply its circles, peopled by universes, passing in proportional segments between the extending symbolical arms of your compass. Suppose now that, from whatever point of the infinite above you, a hand holds another compass or square, then the lines of the celestial triangle will meet of necessity those of the compass of science and will form therewith the mysterious star of Solomon.—E. Lévi, *History of Magic*, English translation by A. E. Waite, p. 83. [Per Mr. E. de St. Q. Isaacson.]

1619. Happy the lot of the pure mathematician. He is judged solely by his peers and the standard is so high that no colleague can ever win a reputation he does not deserve. No cashier writes articles in the *Sunday Times* complaining about the incomprehensibility of Modern Mathematics and comparing it unfavourably with the good old days when mathematicians were content to paper irregularly-shaped rooms or fill bath-tubs with the wastepipe open. Better still, since engineers and physicists have occasionally been able to put his equations to destructive use, he is even given a chair in a State University.—W. H. Auden, " Squares and Oblongs ", an essay published in *Poets at Work*. [Per Prof. Harry M. Gehman.]

MATHEMATICAL SCHOOLBOOKS IN SWITZERLAND.

By E. Barton.

Public education is a cantonal concern in Switzerland, and even the seven universities established in the capitals of the country (Geneva, Basle, etc.) belong to the cantons. The Eidgenössiche Technische Hochschule, called E.T.H., the famous technical university of Zürich, is the only educational institution that belongs to the Swiss Confederation. In Switzerland there are some twenty school systems of various types to suit pupils of different ages, which differ in educational aims and curricula, and which employ no less than three languages of instruction. The great variety of the cantonal schools is further increased by a good many private schools which maintain a very high standard, whose pupils must, on leaving, undergo a final examination followed by an entrance examination before an official board of examiners if they wish to continue their studies at any of the public educational institutions.

In the years up to and during the First World War, in spite of its strong political unity, the decentralization of education in Switzerland and the fact that its languages of instruction were identical with those of its big neighbouring countries, brought about considerable and serious foreign influence in the realm of schoolbooks. Those used in the German-speaking part of the country came from Leipzig, Stuttgart and Berlin; those in the French part from France; while those in use in the Italian part of the country came from Milan and Rome. There were some Swiss-produced textbooks also in use, but these were very few.

This diversity in curricula and schoolbooks involved many disadvantages. If, for example, the parents of a secondary schoolboy had to change their place of residence and as a result the boy had to change schools, he often ran the risk of losing a whole year, since the standard in secondary schools has always been set according to proficiency in mathematics above all else, although of course the pupil has to be accomplished in one or more foreign languages besides his mother tongue. This stress on the importance of mathematics has always prevailed in Switzerland, and it is a constantly-felt difficulty even at the E.T.H. of Zürich, that its undergraduates, coming from the differing secondary school of the cantons, must begin their studies with a very unequal preparatory education. This difficulty is intensified by the fact that the first year's university work leads back to the work done in the last year at school. It is to some slight degree lessened by the function of a federal Commission for leaving certificate examinations. This Commission regulates conditions for medical studies for the whole of Switzerland, laying down also certain principles regarding mathematics, and thus indirectly breaks up to some extent the separatism of the cantons. Its principles are, of course, followed by the secondary schools, since these cannot give up the possibility of their pupils being admitted to the medical faculty. Another step towards uniformity is taken by the annual publication of the mathematical problems set at the Federal Leaving Certificate examinations.

Another and yet more dangerous point, not felt, however, until after the First World War, was the spiritual and to some extent political influence of these schoolbooks of foreign origin. Switzerland, though always open to the intellectual, literary, artistic and scientific trends of neighbouring countries, has always striven to preserve and fortify its political aims and structure, its national independence, and to foster its own economic ends.

When certain political tendencies, appearing first on the southern and later on the northern frontiers of the country, began to spread doctrines and ideas contrary to Swiss outlook and mentality, and when in time

even schoolbooks were affected, it was the mathematics teachers in the German part of the country who first saw the necessity for prompt action. Even schoolbooks of mathematics are not necessarily neutral. Herbart's theory of concentration in its modern application consciously aims at the linking up of the range of subjects taught in order to avoid isolated elements of knowledge. Examples and illustrations are taken in most cases from real life, and provide excellent vehicles for the dissemination of the most widely differing points of view.

That the refusal to comply with these tendencies had been initiated by teachers of mathematics might be due to the fact that the import of mathematical schoolbooks was naturally more considerable than that of books on literature, history or geography ; for the teaching material of these subjects being domestic rather than universal as mathematics, they were taught from Swiss-produced books even in the early years of the century. On the other hand, Basle, where the counteraction started, has a mathematical tradition which stretches back to Euler and the Bernoullis. Generally speaking, Basle has always taken a considerable interest in education, since it has, besides a first-rate humanistic school, an excellent non-classical secondary school and also another secondary school specialising in mathematics and science. (There is but one more school of this type in the country—in Zürich.) And Basle, situated near both French and German frontiers, has for centuries sympathised with the former rather than the latter, in spite of its German-speaking population.

It was in 1928 that the Association of Swiss Teachers of Mathematics (founded in 1901) resolved at the instigation of Prof. Otto Mautz of Basle to have uniform books written and published on all branches of mathematics belonging to the sphere of secondary school-teaching. No doubt this was a daring decision, for we see Brandenberger writing in 1911—on the teaching of mathematics in secondary schools—that it was impossible to think of editing special Swiss schoolbooks in view of the variety of the types of school existent in the country. Now, in 1928, the first thing to be thought of was textbooks, collections of problems and solutions, each part planned in three volumes with separate logarithmic tables. A " Committee of Schoolbooks " was constituted under Prof. Paul Buchner, rector of the Basle Secondary School for Mathematics and Science, and this set to work at once to divide the material for the new scheme among the working sub-committees of German-speaking Switzerland constituted for that purpose.

French Switzerland followed their example much later, although it had already taken part in the work of the German Swiss committees. Tessin and Graubunden (the Italian parts) never joined at all ; this may be accounted for by the fact that Switzerland has no university in which the medium of instruction is Italian, so that students of Italian mother-tongue unless proficient in German or French have always had to attend Italian universities. If, however, they do so, it is only on condition that the textbooks they have used at school have been recognised by Rome.

Managing the financial problems of the undertaking was no easy matter. Since books of foreign origin are printed in far greater numbers of copies, Swiss publishers were naturally not anxious to undertake the publishing of Swiss authors' schoolbooks, the former being cheaper to buy and therefore a more profitable venture ; yet the Association succeeded in persuading the well-known publishers, Orell Füssli of Zürich, to make a contract with them for the whole series planned. They believed that losses would be redeemed by profits, and have not been disappointed in this belief. After some revision certain already proved successes were re-published ; (cf. Dandliker's popular compilation of descriptive geometrical problems). Considerable

reductions were provided for in the event of some of the books being purchased for whole classes in schools. Voellmy, one of the authors, has won round public opinion by a press campaign both persistent and ingenious.

Not all the books projected have been published so far, and there have been no rules laid down as regards the obligatory or exclusive use of the books at issue. Nevertheless, according to statistics known by January 1947, some 180,000 volumes had been sold, representing a sum to the value of 600,000 Swiss francs. In tiny Switzerland this must be looked on as a considerable success springing from the typical Swiss character of the entire scheme and the way it had been put into action. It showed the organising power, perseverance, diligence and carefulness characteristic of the Swiss. Business is discussed by many; attention paid to slightest particulars; mistakes are avoided, for even the smallest details are attended to with precision. The basic principles being agreed upon beforehand, they are then carried through with matter-of-fact conscientiousness.

The venture has undoubtedly been a success. Yet to compile good mathematical schoolbooks is no easy matter, for there is a constant demand for something new, and modern trends of thought cannot be neglected; a middle course has to be struck between strict scientific systematism and the flexibility of teaching; axiomatic construction must be reconciled with intelligible and even interesting lecturing. The compilers of these books have shown due regard for all these points. Since a good teacher likes to have plenty of material, they have arranged the books to contain more than the essential minimum, material which is indispensable being marked in heavy and that which is merely desirable or recommendable in light type.

The arrangement of the complete work in German is as follows:

Algebra (Parts I–IV).
Planimetry (Parts I–II).
Plane trigonometry.
Stereometry.
Descriptive geometry.
Analytic geometry.

(All of them are divided into three parts: textbook, collection of problems, and solutions, the second part being by far the most voluminous.) For so-called higher grade elementary schools a separate collection of algebraical problems and solutions had been published.

Logarithm tables of 5 figures.
Logarithm tables with 4 figures.
Logarithm tables with 5 figures and centigrade division of the right angle.

Most of the thirty volumes originally projected are already in use; some have even reached their seventh edition. There has also appeared recently a *Collection of Physical Problems with Solutions.*

The collective work in French is analogous, but no translations, except for the most recent logarithm book; the books are being written in the original language. There are differences of substance as well as of language; for one, there has been published in French a separate arithmetical work in two volumes in order to include lower school-classes in the uniform book system. There have as yet been but few volumes published: *Géométrie Plane*; *Table de Logarithmes à 4 Décimales*; *Table de Logarithmes à 5 Décimales.* Next to be published is *Algèbre*, then *Trigonométrie.* The latter is to be edited on the most up-to-date principles, based on the vectorial theory.

A more concrete impression of the contents may be gained by taking a closer view of one of these books, say, for instance, the third and fourth volumes of the *Algebra* and the *Logarithm-book.*

The third volume of *Algebra* expands the syllabus of the secondary schools with the elements of mathematical statistics and theory of probability to meet the most recent requirements. When trying to determine the mathematical syllabus, one of the most difficult problems that the secondary school has to face has always been to decide what should be taught of applied mathematics, for not only must the solution be at all times realisable either in figures or diagrams, but the problems while starting from other branches of knowledge must be such as are capable of being dealt with mathematically. To treat statistical problems in mathematics is always simpler, since statistics convey the results of observation only, and it is easier for the pupil to grasp than the abstract calculus. Few adolescent pupils acquire the way of abstract thinking necessary to understand the theory of limits.

This volume deals in due succession with the statistical, arithmetical, geometrical and combinatorial sequences. It includes the elements of compound interest, of annuity and life assurance calculus. Its principal aim is to acquaint the pupil with the bell-shaped curve presenting itself in connection with the problems to be solved, in various aspects; appearing first empirically in relation to statistical sequences, then as the binomial curve of the combinatorial sequences, afterwards as the curve of distribution, and finally as the curve of errors.

Teaching difficulties appear, strictly speaking, only with the theory of probabilities. To overcome the well known difficulties arising from the recent disagreements on the foundations of probability, the material is divided into two parts; in the first part relative frequency and traditional mathematical probability, in the second probability taken in a more recent and stricter sense, are successively dealt with.

The work of Prof. Buchner is exemplified in the fourth volume of the *Algebra*. This eminent mathematician has been the life and soul of the whole scheme, and it is he who has recently published a student's scientific periodical: *Elemente der Mathematik*—"Elements of Mathematics". In his preface he clearly outlines the problems attached to introductory instruction on the elements of differential and integral calculuses, which indeed recur again and again, although not always with the same acuteness, in the most widely varied branches of mathematics. Would it be better to build up to the calculus with systematic groundwork or, on the other hand, to try to convey fundamental concepts and connections by means of practical problems, which may run the risk of seeming a somewhat artificial method?

One extreme point of view was held by the non-classical secondary school of Zürich which had, in the closing decades of the nineteenth century, realised some of these problems; this would effect the transition from the difference-quotient to the differential quotient by geometric means and taking a geometric view. The other was held in Geneva, and would put off certain sets of problems (the solution of equations, for instance) until they came to find a simple and uniform method in the calculus.

Yet it is possible to reconcile the two, for whichever method be employed, in good teaching the fundamentals are identical; cardinal principles must be interpreted by means of object lessons, and linked up with geometrical and mechanical problems. The pupil must be impressed with the profit of principle and method he will derive, the riches of the calculus and its wide application.

The use of logarithms is also a point of conjunction between the followers of the two extremes of mathematical instruction. Those who are used to considering the promotive influence of mathematics upon logical thinking and a philosophic outlook as being the most valuable of its advantages will think it enough to mention the existence of the logarithm tables. Those, on

the other hand, who think the application of mathematics to be at least as important will find great pleasure in Prof. Voellmy's logarithm book, for it contains, besides the mantissas, the auxiliary numbers serving the centigrade division of the right angle, the second powers of the numbers reaching as far as 9999, the square roots, cube roots, natural logarithms of numbers as far as 999, factorials as far as 30, a table of mortality bearing on the population of Switzerland, and several auxiliary tables pertaining to natural science (astronomy, geography, etc). Moreover, it contains an excellent collection of formulae of about thirty pages, which may, together with the logarithm book, be used at examinations for the leaving certificate. This is no doubt a praiseworthy reform, although at the same time memorisation of the formulae ought not to be completely neglected, for readiness of thought in itself will not lead to proper results if it does not ground itself on fixed points, in other words, on formulae ; but the disencumbrance of the memory will at any rate allow for more time and intellectual energy to be expended on the demonstration of formulae and acquiring practice in their application. In this regard Swiss schools consciously deviate from the method of memorisation which formerly prevailed, and which still does in many other countries, notably in France.

The inner unity of the whole series of books published is shown in its uniform axiomatical foundation as well as in its uniform language of formulae. (In this the E.T.H. took a large share, planning and shaping.) So much imagination, as well as concentrated labour, has been put into the scheme that the result cannot but satisfy the most expert and exacting of readers. By it the reputation of Swiss mathematical teaching, although highly appreciated before the movement, cannot but have been enhanced.

Competent Swiss official quarters have made up their minds to aim at similar results in other branches of youth education, in the field of biology, chemistry, philology. Experts in other countries might do worse than turn to good account this pioneer work in the sphere of schoolbook literature.

Geneva-Frontenex, April 1947. E. B.

1620. If we start and build up a logical edifice upon an imprecision, introducing at each stage of the argument other imprecisions, the final result, though appearing irrefragable, may yet be so unsubstantial that the first assault upon the structure brings it to the ground. Perhaps an analogy, introduced from mathematics, may illustrate what is meant. Argument is something like multiplication. Suppose x is taken to represent an idea which it describes with a small amount of imprecision which may be represented by dx. If we use the idea twice over in the steps of an argument, the result in logic is x^2, but the result in reality is $(x+dx)^2$, which is $x^2+2xdx+d^2x$ (*sic*). Now if dx is small, d^2x is negligible and the imprecision is represented by $2dx$. In other words, the " woolliness " of the result is doubled at least. But suppose the woolliness represented by dx was large, d^2x cannot be neglected and $2dx$ becomes formidable.—V. H. Mottram, *The Physical Basis of Personality* (1944 Pelican Edition, pp. 100, 101). [Per Mr. J. B. Douglas.]

1621. A District Commissioner in the West Country closed his instructions to a scout setting out on a " First Class " journey with " Keep all equipment down to a minimum ; remember that 1 oz. multiplied by 14 miles equals approximately 1 ton ".

[Is this because 1 oz. × 14 miles can be expressed as 1 ton by 2 ft.?]

[Per Mr. I. Fitzroy Jones.]

ON DIFFERENTIALS.

By E. G. Phillips.

Mr. D. K. Picken's recent short note * refers the reader to the various articles on differentials which have appeared in the *Gazette* since 1931. On reading it, I was led to refer to the articles which he cites, and I have re-read them all. My own article was the first of these, and I hesitate to intervene again lest I start a second controversy, but I find it difficult to pass without a protest the statement that the definition of the differential dx as the arbitrary infinitesimal Δx is open to serious objection. It also seems to me desirable now to re-affirm the consistency of a theory of differentials which is based on that definition.

The correct definition of an " infinitesimal " is, as the second footnote in Mr. Picken's article states,† " a variable which tends to zero ", but I cannot see how the above definition of the differential dx is at variance with the definition of an infinitesimal. I recall that in my original note ‡ I followed de la Vallée Poussin in deriving the definition of dx from $dy = f'(x)\,\Delta x$, by supposing that $f(x) = x$, but a long experience of teaching differentials, both to beginners and to more mature students, has led me to modify this and use instead the very definition to which Mr. Picken takes exception. I now start by *defining* the differential dx of the independent variable to be the same as the (arbitrary) change Δx in x. §

I then define *differentiability* in the usual way, by saying that $y = f(x)$ is differentiable, if Δy can be expressed in the form

$$\Delta y = A\ \Delta x + \epsilon\ \Delta x,$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Then I define dy to be $A\ \Delta x$ (which is also $A\ dx$). The fact that Δy is not the same as dy is easily illustrated geometrically, || and so it is, in my opinion, quite logical to start teaching differentials in this way. The definition is simple, easily understood and leads to no inconsistency in the subsequent development. Surely that is all we ask of any *definition* in mathematics! Incidentally, it is much easier for a beginner to see the geometrical meaning of dx, Δy and dy than it is for him to understand why $f'(x)$ is the slope of the tangent, which involves the admittedly harder idea of the limiting position of a chord PQ as Q approaches P.

Of course, it is in dealing with functions of more than one variable that the real advantages of the differential method appear. Most of the earlier articles have been mostly concerned with discussions on various ways of *defining* a differential, some of them quite unsuitable as a basis for any theory which is to be presented to immature students. Also the fact that, for functions of one variable, there is no difference between (i) being differentiable and (ii) possessing a derivative, has been rather over-emphasised. My only plea for introducing differentials at all when dealing with functions of one variable is that, since (i) and (ii) are *not* the same for functions of more than one variable, it is an advantage to the student to learn something of the differential at the one-variable stage, for he then finds the idea already familiar when he comes to deal with functions of more than one variable.

It seems to me that, if there is justifiable criticism of the above definition of the differential dx, with, as Mr. Picken says, " the implicit distinction between ' independent ' and ' dependent ' variables ", it strikes at the root

* *Math. Gazette*, XXXI, 252. † XIX, 80. ‡ XV, 401.

§ There is no need in the *definition* for Δx to be *small*, and it is better with beginners to avoid the term " infinitesimal ".

|| See Durell and Robson, *Elem. Calculus*, p. 27 ; or XV, 402.

of the whole theory, for it is just this difference between "independent" and "dependent" variables which has to be kept in mind throughout the whole differential technique. Anyone who has once appreciated the superiority of the differential method over the method of partial derivatives, would not be likely easily to be persuaded to abandon it, unless a really serious, well-substantiated objection could be brought against its rigour. I maintain that this has not been achieved by any of the earlier articles. Most of them seem to be advocating either a more difficult technique or are merely suggesting that there is something "dubious" about differentials!

My references to functions of more than one variable in my first article were slight,* because when I wrote that, I was then thinking more about the schoolmasters' problems in the teaching of the elements of the Calculus. My re-reading of all the earlier articles makes me feel that now a little more attention should be given to the field in which the differential method shows its superiority over any other, viz. in dealing with functions of more than one variable. As there are text-books available † in which this is all fully discussed, I do not want to err on the side of prolixity, but the "doubt" which some of the earlier articles seem to suggest, inspires me to make a few observations, which the "converted" can ignore.

One of the most important features of the theory lies, I believe, in the meaning to be attached to the term "independent variables". When we say that u and v are two of the *independent* variables of any problem, we do not mean simply that u does not depend on v, but that du and dv are constants, or, in other words, $d(du)=d^2u$ and $d(dv)=d^2v$ are both zero. If u is one of the *dependent* variables of the problem, d^2u is not (in general) zero, for du is then a function of the independent variables and can itself be differentiated. When this is understood, and the fundamental theorem on the *permanence of the first differential* has been established, the technique is essentially the same for every problem, whether the variables are given by explicit or implicit functional relations. What I call the fundamental theorem on the permanence of the first differential is the theorem: *If* $z=z(x, y)$ *and* $x=x(u, v)$, $y=y(u, v)$, *then*

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy. \quad \text{.....(1)}$$

If $z=z(x, y)$ and x, y are the independent variables, then (1) is the definition of dz, but the fundamental theorem proves that (1) is still true if x, y are functions of other independent variables u, v. In the first case d^2x and d^2y are zero, but in the second case dx and dy are functions of u, v and can themselves be differentiated.

Example. If three variables x, y, z *are connected by the functional relation* $f(x, y, z)=0$, *and* p, q, r, s, t *have their usual meanings, prove that when* y, z *are the independent variables,*

$$\frac{\partial^2 x}{\partial z^2}=-\frac{r}{p^3}.$$

We start with the permanent equation

$$dz=p\,dx+q\,dy. \quad \text{.....(2)}$$

If we differentiate this, when y, z are the independent variables, we get

$$0=r\,dx^2+2s\,dx\,dy+t\,dy^2+p\,d^2x, \quad \text{.....(3)}$$

* As pointed out in footnote 3 of XIX, 83.

† See, *e.g.*, de la Vallée Poussin's *Cours d'Analyse*, or my *Course of Analysis*.

since dy, dz are now constants. (2) can be rewritten :

$$dx = (dz - q\,dy)/p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and (3) can be rearranged to express d^2x in the form

$$d^2x = A\,dy^2 + 2B\,dy\,dz + C\,dz^2,$$

hence $\partial^2 x/\partial z^2 = C$; and we find, on substituting the value of dx from (4) in (3), that $C = -r/p^3$.

I give this example of the standard procedure to emphasise how the need for distinguishing between the differentials of the dependent and independent variables is fundamental to the whole technique.

My conclusion, then, is this. As the Editor of the *Gazette* pointed out in his historical note,* the rigorous exposition of the differential has been part of the stock-in-trade of Cambridge teachers for at least thirty years. This exposition is, in the main, the same elegant method as that expounded in de la Vallée Poussin's *Cours d'Analyse*. Those who, like Mr. Picken, assert their objection to the definition of the differential dx of the independent variable to be Δx, seem to me to be raising an objection which has the effect of making them unhappy about their fundamentals. It also seems in consequence to lead to a much more difficult and less elegant technique.† But I cannot myself see that the objections they raise to the method which I am advocating are of sufficient importance to cause me to abandon the differential method, which I have advocated and taught for nearly twenty-five years. Nor do I see anything unsound in the development which I follow, and which starts by defining dx to be the same as Δx. E. G. P.

* XVI, 10. † See Picken, XIX, 79 *seq.*

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of **Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby**, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

1622. Harry Bateman, F.R.S., Professor of Mathematics at Pasadena, California, died in February, 1946. . . . He was the most amiable and unworldly of men ; and as he walked slowly along, revolving some mathematical problem, he had the look of a slightly surprised angel.—Obituary notice of Bateman in *British Chess Magazine*, October 1947. [Per Rev. A. F. Mackenzie.]

1623. What is never taken into account here is the fact that a British alphabet of 40 letters would make it possible to save 20 per cent. of time and labour, and that means more than 500,000 per cent. in a year. If that does not make Sir Stafford Cripps as keen a supporter of spelling reform as Mr. Follick, nothing will.—Report of House of Commons debate, *Daily Telegraph*, March 12, 1949. [Per Mr. A. R. Pargeter.]

THE ENERGY OF FLUIDS IN MOTION.

By G. Power.

Problems are constantly occurring in hydrodynamics in which is required the rate of change of kinetic energy within a closed surface drawn in the fluid. It is essential to remember that the expression for this rate of change of energy depends on whether the surface is fixed in space or moving with the fluid. Moreover, a clear definition of the rate of transmission of energy across the surface is necessary. No detailed account of the various cases seems to have been given.

The following is an elementary treatment which summarises briefly the cases which occur.

Compressible Fluid.

Let S be any closed surface in the fluid.

We define the rate of transmission of intrinsic energy of a mass δm of the fluid (or as it is sometimes called, the rate of doing work of the internal pressures of δm) to be

$$p\frac{D}{Dt}\left(\frac{\delta m}{\rho}\right)=p(\nabla\cdot\mathbf{q})\delta v, \quad [\rho\,\delta v=\delta m],$$

with the usual notation. So then I, the total rate of transmission of intrinsic energy in S, is given by

$$I\equiv\int_v p\nabla\cdot\mathbf{q}\,dv,$$

where the integral is taken throughout the volume of S.

We define the rate of transmission of energy across the surface of S to be

$$W\equiv-\int_s p\mathbf{q}\cdot\mathbf{n}\,dS,$$

where the integral is taken over the surface of S, $\mathbf{n}$ being the unit vector in the direction of the outward normal.

The expressions I, W may be put in the scalar form

$$\iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dx\,dy\,dz\,; \quad -\iint p(lu+mv+nw)dS,$$

respectively.

Let T be the total kinetic energy in S, and Q be the rate of work of the external forces $\mathbf{F}$ on the fluid within S. There are two main cases to discuss: (I) S moving with the fluid, (II) S fixed in space.

A boundary surface consisting of moving particles of fluid may be considered as case (I).

(I). This is the case usually discussed in text-books. The rate of change of kinetic energy is $\dfrac{DT}{Dt}$, and it has been shown that

$$\frac{DT}{Dt}=\int_v\rho\mathbf{F}\cdot\mathbf{q}\,dv+\int_v p\nabla\cdot\mathbf{q}\,dv-\int_s p\mathbf{q}\cdot\mathbf{n}\,dS$$
$$=Q+I+W. \quad \text{.....(A)}$$

(II). In this case the rate of change of kinetic energy is

$$\frac{\partial T}{\partial t}=\frac{1}{2}\int_v\frac{\partial\rho}{\partial t}q^2dv+\int_v\rho\mathbf{q}\cdot\frac{\partial\mathbf{q}}{\partial t}dv.$$

But $$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \quad \text{(equation of motion).}$$

Thus $$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p.$$

Thus $$-\nabla p = -\rho \mathbf{F} + \rho \frac{\partial \mathbf{q}}{\partial t} + \rho \mathbf{q} \cdot \nabla \mathbf{q},$$

and so $$-\nabla p \cdot \mathbf{q} = -\rho \mathbf{F} \cdot \mathbf{q} + \rho \mathbf{q} \cdot \frac{\partial \mathbf{q}}{\partial t} + \rho \mathbf{q} \cdot \nabla \mathbf{q} \cdot \mathbf{q}.$$

Now $$W = -\int_s p\mathbf{q} \cdot \mathbf{n}\, dS = -\int_v \nabla \cdot (p\mathbf{q})\, dv$$

$$= -\int_v (\nabla p \cdot \mathbf{q} + p \nabla \cdot \mathbf{q})\, dv = -\int_v \nabla p \cdot \mathbf{q}\, dv - I.$$

Hence $$W + I = -\int_v \nabla p \cdot \mathbf{q}\, dv = -Q + \frac{\partial T}{\partial t} - \frac{1}{2}\int_v \frac{\partial \rho}{\partial t} q^2 dv + \int_v \rho \mathbf{q} \cdot \nabla \mathbf{q} \cdot \mathbf{q}\, dv.$$

Thus $$\frac{\partial T}{\partial t} = Q + I + W + \frac{1}{2}\int_v \frac{\partial \rho}{dt} q^2 dv - \int_v \rho \mathbf{q} \cdot \nabla \mathbf{q} \cdot \mathbf{q}\, dv. \quad \text{.....(B)}$$

As $\mathbf{q} \cdot \nabla$ is the scalar operation $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, the last term in the expression for $\frac{\partial T}{\partial t}$ may be put in the scalar form

$$\iiint \rho q \left(u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}\right) dx\, dy\, dz.$$

From case (II) we can deduce the rate of change of kinetic energy within S, when S is moving with uniform velocity $\mathbf{r}$.

We require, in this case,

$$\frac{\partial}{\partial t}\int_v \tfrac{1}{2}\rho(\mathbf{q} + \mathbf{r})^2 dv = \tfrac{1}{2}\int_v (\mathbf{q} + \mathbf{r})^2 \frac{\partial \rho}{\partial t}\, dv + \int_v \rho(\mathbf{q} + \mathbf{r}) \cdot \frac{\partial \mathbf{q}}{\partial t}\, dv \quad [\mathbf{r} \text{ is constant}]$$

$$= \tfrac{1}{2}\int_v (\mathbf{q} + \mathbf{r})^2 \frac{\partial \rho}{\partial t}\, dv + \int_v \rho \mathbf{q} \cdot \frac{\partial \mathbf{q}}{\partial t}\, dv + \int_v \rho \mathbf{r} \cdot \frac{\partial \mathbf{q}}{\partial t}\, dv,$$

where, as before,

$$\int_v \rho \mathbf{q} \cdot \frac{\partial \mathbf{q}}{\partial t}\, dv = W + I + Q - \int_v \rho \mathbf{q} \cdot \nabla \mathbf{q} \cdot \mathbf{q}\, dv,$$

and $$\int_v \rho \mathbf{r} \cdot \frac{\partial \mathbf{q}}{\partial t}\, dv = \int_v \mathbf{r} \cdot [\rho \mathbf{F} - \nabla p - \rho \mathbf{q} \cdot \nabla \mathbf{q}]\, dv,$$

where $\mathbf{r}$ is the velocity of S, and $\mathbf{q}$ represents velocity referred to axes moving with velocity $\mathbf{r}$.

Other results may be obtained in a similar manner.

Incompressible Fluid.

From the equation of continuity we see immediately that $I = 0$, and so the equating of the rate of change of kinetic energy to the rate of work done by the pressure on the external boundary is justified when the boundary is moving with the fluid provided that the rate at which the external forces work is zero. If ρ is not constant, the rate of work done by the internal pressures must be taken into account. These are well-known results.

If the surface is fixed in space, and the fluid (incompressible) satisfies the condition for the continued existence of a velocity potential ϕ, then the rate of change of kinetic energy is

$$\tfrac{1}{2}\rho\frac{\partial}{\partial t}\int_v q^2 dv = \tfrac{1}{2}\rho\frac{\partial}{\partial t}\int_v \Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 dv$$

$$= \rho\int_v \Sigma\frac{\partial\phi}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)dv = \rho\int_v \Sigma\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)dv$$

$$= \rho\int_v \Sigma\frac{\partial\phi}{\partial x}\frac{\partial}{dx}\left(\frac{\partial\phi}{\partial t}\right)dv + \rho\int_v \frac{\partial\phi}{\partial t}\left(\Sigma\frac{\partial^2\phi}{\partial x^2}\right)dv$$

[Since this latter term vanishes].

$$= \rho\int_s \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial n}\,dS,$$

where ∂n is an element of normal drawn out of the surface S. An often-quoted example (Ramsey, Weatherburn, etc.) gives this expression to be the rate at which energy flows across the boundary of S, which is, in fact, the W of the formula (B).

Viscous Fluids.

For viscous fluids, we define W to be $\int_s \mathbf{n}\,.\,\Psi\mathbf{q}\,dS$, where Ψ is the stress dyadic.

The formulae (A) (B) then remain essentially the same except for a term D, representing the rate of dissipation of energy which has to be subtracted from the right-hand side.

Thus we see that the rate of dissipation of energy may be equated to the rate of working of the stress over a surface moving with the fluid, when $I = 0$ (fluid incompressible), $Q = 0$, and the motion such that $\frac{DT}{Dt}$ vanishes or may be neglected.

G. Power.

1624. The process of testing Mersenne's numbers individually has no subtlety or mathematical interest; it is done by the use of "the Sieve of Eratosthenes", a grim grind of successive long division. But as every prime number up to the square root of the number tested has to be tried, and the square root in this case has nearly forty digits, the operation has occupied generations. It has, however, been recently completed, and Mersenne turns out to have been right.—*The Times*, June 14, 1949. [Per Mr. A. Robson.]

1625. Sir,—Relationship between the area of a circle and the area of a square of equal length of side to the diameter appears to be exact, and is consequently a straight horizontal graph.

Based on 3·1416 the difference is ·2146, or based on 3·14592 the difference is ·214602. In other words, the area of any circle is ·7854 of the square in which it fits.

Since the area of a square can be proved exactly and proof (pi) involves abtruse mathematics with hundreds of decimals, is it not time the physicists proved the mathematicians as in these days of advanced science they should be well able to do? I suggest that the physicists give us a standard circle and sphere from which all others may be proved by simpler formula, in addition to our standard lineal measure.

The length of the sides of squares and the circumference of relative circles show straight line graphs, as also the volumes of spheres to each other seem to do.—Letter to *Western Morning News*, July 2, 1947. [Per Mr. D. Ralph.]

MATHEMATICAL NOTES.

2057. *Sign in relation to curvature* (XXVII, No. 274, p. 77, May, 1943, and XXVIII, No. 282, p. 185, Note 1754, December, 1944).

This is a note following upon Robson (XXX, No. 288, pp. 1–5, February 1946) and the article on " Sign . . ." by the present writer (XXX, No. 291, October, 1946 : see p. 208).

1. The resolving principle, in these questions of sign, appears to be clear-cut discrimination between a quantity (here, geometrical : *length*, *curvature*, etc.) and the real number which is its " measure ".*

Sign is essentially characteristic of real number ; it is not necessarily characteristic of quantity other than number. *Measure*-numbers are relative to a framework—of measurement and other reference ; and the element of sign, in the analytical treatment of geometry, is generally significant, in that relationship. But the facts are not quite so straightforward in the case of curvature (because of its " intrinsic " character) as in those discussed in the October, 1946, article : that is what makes the February, 1946, article so valuable, in the context.

2. *Curvature.* Starting out from the definition : the *curvature* of a (finite†) arc PQ is a quantity jointly proportional, directly to the " *angle* of bend ",‡ inversely to the *length* of the arc, we get the form $k = a/l$, (say), which has reference to the " derived unit " of *curvature* (corresponding, in the proportion, to *unit length* and *unit angle* §) : and, if *circular measure* of angle be used, this unit of curvature is the curvature of unit circle.‖ Hence, again, the fundamental analytical form $\kappa = d\psi/ds$, for the measure of curvature-at-a-point on a curve—using a standard notation ; but it is to be noted that the actual " frame of reference " for s and ψ is not involved in this expression for κ.¶ And, with the obvious correlation of the sign of ds with the ψ-direction, the *number* κ is positive or negative according as the " onward tangent " is turning positively or negatively in the plane of the curve, i.e., according as the curve lies to the " left " or to the " right " of its onward tangent (in the neighbourhood of the point in question)—in terms of the standard definition ** of sign of angles in plane geometry.

This criterion, it is easy to see, covers the whole ground. In so far as the specification of the onward tangent is continuous—say, in terms of s itself, or of a parameter, t—κ varies continuously ; but discontinuity in the specification

* For a full discussion of this question, see XXVI, No. 269 (May, 1942), pp. 87–95.

† It is customary to use the term " *mean* " (or *average*) *curvature* for the *finite* arc (and, so, in analogous cases, such as that of *speed*). But that is to " put the cart before the horse ! " ; the finite interval is always basic, though the limiting infinitesimal consequence is, in general, the more important.

‡ Or some such phrase for the *angle* in question. To call *it* " the total (or integral) *curvature* " is to aggravate the prevailing confusion between quantities of different kinds.

§ See §§ 5–7 of the May, 1942, article (cited above).

‖ The curvature of all arcs of a given circle being the same—determined by $k = 1/r$ (a positive or negative number according to the sense in which the " onward tangent " turns).

¶ Nor is any functional *expression* for ψ in terms of s (or inversely) implied. (The discussion of " the differential quotient " in XIX, No. 233, pp. 80–81 (May 1935) is relevant.)

** Mr. Robson's point about " definitions and conventions " (XXX, p. 1) is important. On the whole, " definition " gives the better general line of approach—if we mean by that word the giving of definite formulation to what is plainly significant (not merely conventional). Definitions in general are by no means arbitrary.

of the tangent is reflected in discontinuity in the variation of κ : *not in that of the curvature itself*, which is of course independent of the framework to which κ has reference. It is of interest to look, from this point of view, at the two standard types of case in which such discontinuity arises (XXX, p. 2).

3. (i) For the Cartesian form $y=f(x)$—taken, basically, as one-valued—

$$\tan\psi = dy/dx \text{ gives } \sec^2\psi \,.\, \kappa = (d^2y/dx^2) \,.\, (dx/ds), \text{ or } \sec^3\psi \,.\, \kappa = d^2y/dx^2 ;$$

and the sign of κ is, therefore, that of y'' (since determination of the " onward tangent " by *increase* of x gives sec ψ positive).

For the general Cartesian form, discontinuity is involved in the many-valuedness of y—at the " branch " extremes.

(ii) For the Polar form $r=f(\theta)$—taken, again, as basically one-valued—

$$d\psi = d\theta + d\phi \text{ * and } \cot\phi = (dr/d\theta)/r,$$

give
$$r^2 \operatorname{cosec}^2\phi \,.\, \kappa = \{r^2 + 2(dr/d\theta)^2 - r \,.\, d^2r/d\theta^2\} \,.\, (d\theta/ds)$$

and thence
$$r^3 \operatorname{cosec}^3\phi \,.\, \kappa = r^2 + 2(dr/d\theta^2) - r \,.\, d^2r/d\theta^2 ;$$

and, here again, the ambiguity of sign in the factor cosec ϕ (when expressed in terms of θ) is resolved (as positive) from the determination of the " onward tangent " by *increase* of θ.

Many-valuedness involves discontinuity of κ (at the points of contact of tangents from O).

4. *Circle of curvature*, etc. This is *the circle which touches the curve*, at P, *and has the same* κ *there* (by reference to the same onward tangent) ; *its radius*, determined by $\rho = 1/\kappa$, " *the radius of curvature* " ; *its centre*, C, " *the centre of curvature* ".†

The Cartesian coordinates of C are, thence,

$$x + \rho \,.\, \cos(\psi + \tfrac{1}{2}\pi) \quad \text{and} \quad y + \rho \,.\, \sin(\psi + \tfrac{1}{2}\pi)$$
$$= x - \rho \,.\, \sin\psi \qquad\qquad = y + \rho \,.\, \cos\psi,$$

forms which are *general* (in terms of the sign of $\rho = 1/\kappa$, as in §2, above).

Using §3 (i), these give the standard Cartesian expressions (as in XXVII, p. 78).

5. *The pedal variable*, p, etc. (i) If we lay emphasis, explicitly on the fact that p denotes, *not the perpendicular distance* (say OT) *itself*, but a *measure* of that distance, the question of its sign becomes a secondary consideration ; and we may properly *define* p and t (the measure of TP) in the old way (from the simple standard figure, in which the θ, ϕ and ψ angles are all positive acute), viz., by

$$p = r \,.\, \sin\phi \quad \text{and} \quad t = r \,.\, \cos\phi,$$

thus throwing the onus of sign upon r and ϕ (as already defined. And, if we then use p, χ for the Polar coordinates of T, this definition amounts to choosing $\chi = \psi - \frac{1}{2}\pi$ (of the two possibilities).

(ii) From these, we readily obtain ‡

$$dp = t \,.\, d\psi = r \,.\, \kappa \,.\, dr$$

(whence $t = dp/d\psi$ and $\rho = r \,.\, dr/dp$) and

$$dt = ds - p \,.\, d\psi$$

* This *differential* relation—unlike that between the parent variables—is unambiguous.

† This, surely, is the right way to define these terms (see 5, below).

‡ As in XIX, No. 233, p. 82. Here, again, no *functional* expressions for the relations between the variables need be implied.

(whence $\rho = p + dt/d\psi = p + d^2p/d\psi^2$)—general *analytical* forms, readily seen to correspond to the limiting forms of the *geometrical* relations

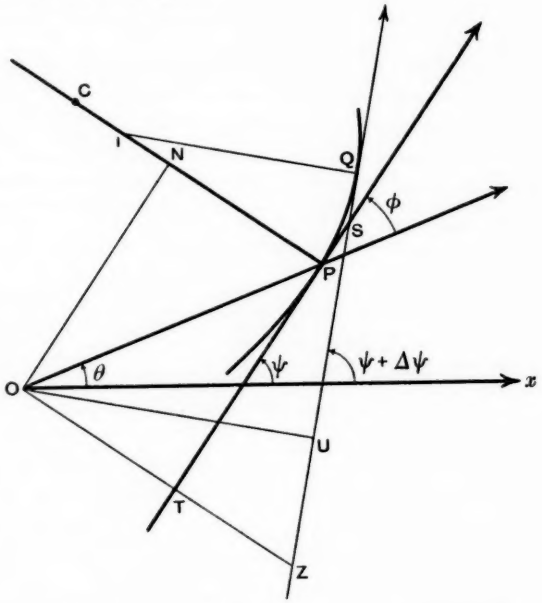
$$OZ - OT = TZ = \tan \Delta\psi \,.\, ST$$

and

$$SZ - SU = UZ = \tan \Delta\psi \,.\, OU$$

$$= \sec \Delta\psi \,.\, (TP + PS) - (UQ - SQ)$$

with notation as in the diagram.*



(iii) The type of fact associated with this diagram (and other diagrams of the kind) are in the older books—but scattered, and not in quite the right perspective ; † in particular, the very interesting details (as in XXVIII, Note 1754) of the successive differentiation (with respect to χ) of the Cartesian straight line form

$$\cos \chi \,.\, x + \sin \chi \,.\, y = p$$

* This " pure geometry " is quite subordinate in the mathematical argument ; and questions of rigour in proof—and of the exceptional cases—are beside the point. The onus of *proof* is on the analytical discussion.

The figure has again been drawn for me by Mr. J. M. Allen (see XXX, pp. 204–5). On this type of " simplest figure which exhibits all the essential facts ", there is a relevant short note, written a quarter of a century ago (XI, No. 161, pp. 188–9) on " geometrical generality ", which holds good.

† *E.g.* Joseph Edwards and Horace Lamb. There has been some considerable loss in the swing of the pendulum, from intuition to rigour ; and a tendency to sacrifice naturalness of approach—which sacrifice, in fact, is hardly ever necessary to rigorous treatment. Both can generally be conserved.

—the significance of which, in the present view, is that they give a simple analytical proof, in terms of (ii) above, of the standard proposition that "*the ultimate intersection of consecutive normals*" *is the centre of curvature* (as defined in §4 above).

(iv) And, in this order of ideas, the other important proposition of this group, viz : *That the circle of curvature is the osculating circle*, may be regarded as a more or less obvious corollary. D. K. PICKEN.

2058. *A property of the cyclic quadrilateral.*

A well-known geometrical problem states that if the opposite sides of a cyclic quadrilateral are produced to meet, the bisectors of the angles so formed are perpendicular. Numerous proofs exist, from Casey's which uses the properties of arcs to various trigonometrical or analytical demonstrations : one very neat proof derives from equiangular and isosceles triangles. The following carries the question a stage further.

If $ABA'B'$ be a cyclic quadrilateral in which BA, $A'B'$ meet at C' and $B'A$, $A'B$ meet at C, the bisectors of the angles C, C' meet at right angles at a point lying on the line joining the mid-points of the diagonals, and if the quadrilateral is also circumcyclic this point is the centre of the in-circle.

Let $CA=a$, $CB=b$, $CA'=a'$, $CB'=b'$ so that $ab'=a'b$, and let the mid-points of BB', AA' be M and N respectively : let also the bisector of $\angle C$ meet AB at P and $A'B'$ at Q. At A, B, A', B' place particles respectively proportional to ba', aa', ab', aa'. The particles at B, B' are equivalent to $2aa'$ at M, and those at A, A' to $2ba'$ at N, so that the centroid G lies on MN and divides it in the ratio $b:a=CB:CA$. Moreover, the particles at A, B are equivalent to $(a+b)a'$ at P since $AP:PB=AC:BC=a:b$, and those at A', B' are equivalent to $(b'+a')a=(a+b)a'$ at Q.

Thus the centroid G also bisects PQ and so lies on the bisector of $\angle C$.

Since $\angle CBA$ is supplementary to $\angle C'B'A$, so that $C'B':C'A=CB:CA$, the same weighting shows that G lies on the bisector of $\angle C'$. Also since $\angle C'GP$ and $\angle C'GQ$ are supplementary, $\angle C'PG$ and $\angle C'QG$ cannot be supplementary and so are equal. Hence $\angle C'GC=\angle C'GQ=$ right angle. B. A. SWINDEN.

2059. *Loss of energy on impact.*

In view of the methods of finding the loss of kinetic energy in direct impact still appearing in textbooks, I venture to suggest that it might be of advantage if the following were more generally known.

Using the usual notation and taking R as the magnitude of the impulse,

$$R=m_1u_1-m_1v_1=m_2v_2-m_2u_2, \quad \text{.....(1)}$$

$$v_2-v_1=e(u_1-u_2). \quad \text{.....(2)}$$

Loss of kinetic energy

$$\begin{aligned}&=(\tfrac{1}{2}m_1u_1^2-\tfrac{1}{2}m_1v_1^2)-(\tfrac{1}{2}m_2v_2^2-\tfrac{1}{2}m_2u_2^2)\\&=\tfrac{1}{2}m_1(u_1-v_1)(u_1+v_1)-\tfrac{1}{2}m_2(v_2-u_2)(v_2+u_2)\\&=\tfrac{1}{2}R(u_1+v_1-v_2-u_2)\\&=\tfrac{1}{2}R(1-e)(u_1-u_2), \quad \text{by (2).}\end{aligned}$$

From (1) and (2),

$$\begin{aligned}R/m_1m_2&=(u_1-v_1)/m_2=(v_2-u_2)/m_1\\&=(u_1-v_1+v_2-u_2)/(m_1+m_2)\\&=(1+e)(u_1-u_2)/(m_1+m_2),\end{aligned}$$

whence loss of kinetic energy

$$=\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

The expression $\frac{1}{2}R(u_1 + v_1 - u_2 - v_2)$ gives the loss of kinetic energy as one-half the product of the impulse and the loss in magnitude of relative velocity, suggesting that instead of " coefficient of restitution " perhaps " coefficient of loss " would be better. The recurrence of the expression $(1 - e)$ in various problems on impact tends to confirm this. W. H. E. BENTLEY.

2060. *The vector triple product.*

In his *Vectorial mechanics* E. A. Milne, after giving three different proofs of the fundamental triple product formula

$$[\mathbf{R}[\mathbf{PQ}]] = (\mathbf{RQ})\mathbf{P} - (\mathbf{RP})\mathbf{Q},$$

states that " if we attempt to proceed by resolving $\mathbf{R}$ into components along $\mathbf{P}$ and $\mathbf{Q}$ we encounter difficulties ". In view of this statement readers may be interested in the following proof which proceeds in precisely this way.

The fundamental definitions of the scalar and vector products imply that $[\mathbf{P}[\mathbf{QP}]]$ and $P^2\mathbf{Q} - (\mathbf{PQ})\mathbf{P}$ both represent P^2 times the component of $\mathbf{Q}$ perpendicular to $\mathbf{P}$ in the plane of $\mathbf{P}$ and $\mathbf{Q}$, and are therefore equal. We write then

$$\mathbf{R} = \lambda\mathbf{P} + \mu\mathbf{Q} + \nu[\mathbf{PQ}]$$

where $$(\mathbf{RP}) = \lambda P^2 + \mu(\mathbf{PQ}), \quad (\mathbf{RQ}) = \lambda(\mathbf{PQ}) + \mu Q^2.$$

Then
$$\begin{aligned}[\mathbf{R}[\mathbf{PQ}]] &= \lambda[\mathbf{P}[\mathbf{PQ}]] + \mu[\mathbf{Q}[\mathbf{PQ}]] \\ &= \lambda\{(\mathbf{PQ})\mathbf{P} - P^2\mathbf{Q}\} + \mu\{Q^2\mathbf{P} - (\mathbf{QP})\mathbf{Q}\} \\ &= (\mathbf{RQ})\mathbf{P} - (\mathbf{RP})\mathbf{Q}.\end{aligned}$$

This proof seems more concise than any given by Milne and concedes nothing in generality. G. H. LIVENS.

2061. *A geometrical proof of a problem in algebra.*

A past question in a London University examination paper reads :

" In a determinant C of order n the element in the jth row and kth column is c_{jk}. If γ_{jk} is the co-factor of c_{jk} in C, and Γ is the determinant whose elements are γ_{jk}, prove that

$$\Gamma = C^{n-1}.$$

If the elements c_{jk} are real and subject to the condition

$$\sum_{k=1}^{n} c_{jk}^2 = 1, \quad (j = 1, 2, \ldots, n)$$

show that $$-1 \leqslant C \leqslant 1.$$"

The following geometrical proof of the second part may be of interest.

If we consider each row of C as the coordinates of a point in n-dimensional Cartesian geometry, then since $\Sigma c_{jk}^2 = 1$ each point is at unit distance from the origin. With the appropriate sign, C is the content of the parallelotope with the lines joining the n points to the origin as adjacent edges. The content is clearly a maximum (in absolute value) when the edges are mutually perpendicular, and its value is then 1. Hence $-1 \leqslant C \leqslant 1$.

An analytical proof is as follows. Using Lagrange's method of undetermined multipliers, we have to choose λ_j $(j = 1, 2, \ldots, n)$ so as to make the partial derivatives of

$$T = |\, c_{jk} \,| - \tfrac{1}{2} \sum_{j=1}^{n} \lambda_j \left(\sum_{k=1}^{n} c_{jk}^2 - 1 \right)$$

with respect to the c_{jk} all zero, in order to find the turning values of C.

Now $\partial T/\partial c_{jk} = \gamma_{jk} - \lambda_j c_{jk} \quad (j = 1, 2, \dots, n\,;\ k = 1, 2, \dots, n).$

Hence $\gamma_{jk} = \lambda_j c_{jk} \quad (j = 1, 2, \dots, n\,;\ k = 1, 2, \dots, n).$

Keeping j fixed, and multiplying the equations corresponding to $k = 1, 2, \dots, n$, by $c_{lk} (k = 1, 2, \dots, n)$ $(l \neq j)$, we have

$$\lambda_j \sum_{k=1}^{n} c_{jk} c_{lk} = 0 \quad (j = 1, 2, \dots, n\,;\ l \neq j).$$

Now if any λ_j is zero, γ_{jk} is zero for all k from 1 to n, that is, Γ has one row zero and, since $\Gamma = C^{n-1}$, $C = 0$.

We can therefore assume that no λ_j is zero, in which case

$$\sum_{k=1}^{n} c_{jk} c_{lk} = 0 \quad (j = 1, 2, \dots, n\,;\ l \neq j).$$

Hence C^2, obtained by multiplying C by itself by rows, has 1 down the leading diagonal and 0 everywhere else; thus $C^2 = 1$, $C = \pm 1$. These are the maximum and minimum values of C, and so $-1 \leqslant C \leqslant 1$. E. F. G. PRIMROSE.

2062. *A proof of the product rule for determinants.**

If $a = || a_{ij} ||$, $b = || b_{ij} ||$ are two determinants of order n, the familiar product rule tells us that $ab = c = || c_{ij} ||$, where

$$c_{ij} = a_{i1} b_{j1} + a_{i2} b_{j2} + a_{i3} b_{j3} + \dots + \dots = a_{ik} b_{jk}, \quad \dots\dots\dots\dots(1)$$

the repeated index indicating, as usual, summation from 1 to n.

The standard proofs of this rule have always seemed to the writer to be unduly complicated, considering the rather obvious form of the result.

The following proof may be of interest to *Gazette* readers. Let A_{ij}, B_{ij}, C_{ij} be the co-factors in a, b, c respectively. Then, according to the familiar expansion rules, we have

$$a_{ik} A_{jk} = a\delta_{ij}, \quad a_{ki} A_{kj} = a\delta_{ij}, \quad \dots\dots\dots\dots(2)$$

and similar results for b and c, where, as usual,

$$\delta_{ij} = 1 \quad (i = j)\,; \qquad \delta_{ij} = 0 \quad (i \neq j). \quad \dots\dots\dots\dots(3)$$

Thus $c_{il}(A_{jm} B_{lm}) = a_{ik} b_{lk} A_{jm} B_{lm} = b\delta_{km} a_{ik} A_{jm} = b a_{im} A_{jm} = ab\delta_{ij}. \dots\dots\dots(4)$

But, as in (2), $c_{il} C_{jl} = c\delta_{ij}. \quad \dots\dots\dots\dots(5)$

Comparing (4), (5) we see that

$$C_{jl} = \frac{c}{ab} A_{jm} B_{lm}, \quad \dots\dots\dots\dots(6)$$

since these n^2 quantities satisfy the same n^2 equations.

Now C_{11} involves only those c_{ij} with $i, j \neq 1$, and therefore, from (1), does not involve the elements a_{1k}, b_{1k}, *i.e.* the elements in the first rows of a, b. This is also true for the quantities A_{1m}, B_{1m}. Hence $\dfrac{C_{11}}{A_{1m} B_{1m}}$ does not involve the elements a_{1k}, b_{1k}, and therefore, from (6), $\dfrac{c}{ab}$ must be invariant under any change of the elements in the first rows of a, b. Extending this argument (considering successively C_{22}, C_{33}, etc.) we see that $\dfrac{c}{ab}$ is completely indepen-

*Mr. Walker's note was received before Note 2017 was in print. The ideas are similar but the different line of reasoning justifies the appearance of a second Note on this topic.—Editor.

dent of the elements of either a or b. To see that $\frac{c}{ab}=1$, we have only to consider the particular case $a_{ij}=b_{ij}=\delta_{ij}$, so that $a=b=c=1$.

A. W. WALKER.

2063. *A question of hats.*

A teacher of Mathematics can nowadays find, scattered about the many puzzle-problem books that are so popular, problems which can be used to add a spice of amusement when dealing with certain mathematical processes —especially equations of all sorts. The following has, I think, some interest and is within the scope of Scholarship candidates. The question is taken from *Something to think about* by Hubert Phillips (Ptarmigan books).

" Four men dined at the Ambrosia. They parked their hats in the cloak-room. Each of the four diners came away wearing a hat belonging to one of the others. In how many ways could this have happened? "

It is not difficult to see that the answer is 9. But the mathematician will naturally wish to discover the general rule. What will be the answer for n men? Call the number of ways u_n, and the diners $A, B, C, \ldots$. If A interchanges hats with B, the remaining $n-2$ can each wear a wrong hat in u_{n-2} ways. The result is the same if A interchanges with C. Therefore if A interchanges with *one* of the others the number of ways is covered by $(n-1)u_{n-2}$. If A wears B's hat but B does not wear A's, the number of ways is u_{n-1} (for, as far as B is concerned, A's hat is barred to him, this possibility having already been allowed for). So we get $(n-1)u_{n-1}$ more ways. This exhausts the possibilities. So we have the equation :

$$u_n=(n-1)(u_{n-1}+u_{n-2}).$$

This is a well-known type of difference equation, though it is not easy to come across illustrations of its use.

It can be written :

$$u_n-nu_{n-1}=-(u_{n-1}-\overline{n-1}\,u_{n-2}).$$

Stepping down we have :

$$u_{n-1}-\overline{n-1}\,u_{n-2}=-(u_{n-2}-\overline{n-2}\,u_{n-3}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

down to $u_4-4u_3=-(u_3-3u_2)$.

From the conditions of the question, $u_2=1$, $u_3=2$. Hence $u_3-3u_2=-1$. So we have $u_n-nu_{n-1}=(-1)^n$.

This may be written :

$$\frac{u_n}{n!}-\frac{u_{n-1}}{(n-1)!}=\frac{(-1)^n}{n!}.$$

So

$$\frac{u_{n-1}}{(n-1)!}-\frac{u_{n-2}}{(n-2)!}=\frac{(-1)^{n-1}}{(n-1)!},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{u_3}{3!}-\frac{u_2}{2!}=\frac{(-1)^3}{3!},$$

$$\frac{u_2}{2!}=\frac{1}{2!}.$$

Whence

$$u_n=n!\left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\ldots\ldots(-)^n\frac{1}{n!}\right\}.$$

(The solution on these lines of a similar equation will be found in *Advanced Algebra*, Durell and Robson, Vol. II, p. 232.)

We can calculate values of u_n from this formula. But for small values of n it is simpler to use the equation itself, *i.e.*

$$u_n = (n-1)(u_{n-1} + u_{n-2}).$$

Thus :

$$\begin{aligned}
u_2 &= 1 \\
u_3 &= 2 & u_2 + u_3 &= 3 \\
u_4 &= 9 & u_3 + u_4 &= 11 \\
u_5 &= 44 & u_4 + u_5 &= 53 \\
u_6 &= 265 & u_5 + u_6 &= 309 \\
u_7 &= 1854 & u_6 + u_7 &= 2119 \\
u_8 &= 14833 & u_7 + u_8 &= 16687 \\
u_9 &= 133496 & u_8 + u_9 &= 148329 \\
u_{10} &= 1334961
\end{aligned}$$

(The reason why the first six digits of u_{10} should be those of u_9 is a nice teaching point.)

As $n \to \infty$, the series in the bracket of the expression for $u_n \to e^{-1}$. Thus for large values of n, $\dfrac{n!}{u_n} \to e$.

Testing this for $n = 10$. $10! = 3{,}628{,}800$.

Using three-figure accuracy and slide-rule $\dfrac{10!}{u_{10}}$ comes to be $\doteq 2{\cdot}72$.

(*Note.* A similar problem to this was attacked by Mr. R. L. Goodstein in *M.G.*, Vol. XXIII, pp. 205, 206, Note 1363. He considers the case of n objects and n holes in which they can be put. He shows that the number of arrangements in which no object is in its right hole is u_n, as given above. He uses an argument based on Combinations. His problem was this : " Two shuffled packs of cards are placed side by side downwards. Two cards, one from each pack, are drawn repeatedly, no card being replaced. To find the chance that at the same draw the two cards are of the same suit and denomination." The chance is $1 - e^{-1}$. With an error of less than 1/53!, this is roughly 17/27.)

H. E. PIGGOTT.

2064. *On Notes* 1542, 1801, 1843.

In connection with the differential equation satisfied by y, where

$$y^3 + 3xy + 2x^3 = 0,$$

it is interesting to note that the absence of the y^2 term necessarily implies a *homogeneous* differential equation. For, let y_1, y_2, y_3 be the roots of the cubic ; then $y_1 + y_2 + y_3 = 0$, which is not possible for solutions of an inhomogeneous differential equation. The same reasoning applies to the more general case of a polynomial of degree n with the y^{n-1} term missing.

A. W. WALKER.

2065. *Reversion of numbers.*

The property mentioned by S. Parameswaran in Note 1958, May, 1947, p. 110, about the difference of two numbers one of which is the other reversed suggests a similar property when the numbers are replaced by their squares.

If a and b are two groups each of k digits, then

$$\overline{ab}^2 - \overline{ba}^2 = (a^2 - b^2)(9)_{2k},$$

where $(9)_n$ is 999 ... to n digits.

For instance $$32^2 - 23^2 = 5 \cdot 99,$$

$$1312^2 - 1213^2 = 25 \cdot 9999.$$

Both properties are also true in a numeral system with any basis A ; then

$$\overline{ab} - \overline{ba} = (a - b)(A - 1)_k,$$

$$\overline{ab}^2 - \overline{ba}^2 = (a^2 - b^2)(A - 1)_{2k},$$

where $(A - 1)_n$ is $(A - 1)(A - 1)(A - 1) \ldots$ to n digits. R. GOORMAGHTIGH.

2066. *The chameleon.*

The conic $$x^2 - 2ixy - y^2 = 1$$

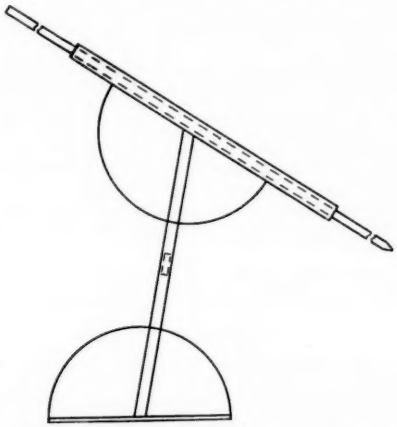
satisfies the conditions for (i) a parabola, (ii) a rectangular hyperbola, (iii) a circle, in the form that $(a - b)^2 + 4h^2 = 0$, showing that its principal axes are equal, and (iv) a pair of straight lines.

On the other hand, it represents the point-pair $(\pm 1, 0)$.

H. V. MALLISON.

2067. *The conesector.*

The instrument was designed primarily for drawing conic sections, but it has been found that it is too inaccurate, due to the flexibility of the metal used, to be of great value in this application. It offers, however, a very clear method of demonstrating the formation of conic sections.



With the instrument, any section of any cone, at a given distance along the axis from the vertex, may be drawn.

The cone is made by a generator, rotating about an axis of fixed length, so that the angle of the generator to the axis, and the axis to the base plane may be varied at will. The generator consists of a sleeve and a rod working in it.

By variation of the angles of the cone and base very fair representations have been obtained of ellipses, parabolae, hyperbolae, and even the extreme case of a straight line.

R. T. TUGMAN.

REVIEWS.

Divergent Series. By G. H. HARDY. Pp. xvi, 396. 30s. 1949. (Geoffrey Cumberlege, Oxford University Press)

This book by the late Professor Hardy contains the matter of his courses of lectures delivered in Cambridge at intervals between 1931 and his death in 1947, revised and enlarged in a form suitable for publication. The subject is one in which, for many years, he had displayed an absorbing interest, and his many important contributions to its development, both alone and in collaboration with Littlewood, have rightly led pure mathematicians in all parts of the world to regard him as its undisputed master. As all his friends and pupils well knew, Hardy possessed a rare gift, the ability of a great mind to convey some of his greatness to others less gifted than himself, by means of exceptionally lucid exposition, both orally and in writing. In the most intricate mathematical analysis he was able to guide his audience through the difficulties in a manner that made them appear almost disarmingly simple, so much so that many a student has confessed to having attended the same course of lectures twice, for the sheer joy of watching once again his master's superb technique and expository skill.

It is no exaggeration to say that the theory of *Divergent Series* illustrates modern analysis in one of its most intricate (and at the same time one of its most attractive) forms, and anyone who has occasion to read or merely to refer to this book will at once see the force of the remarks in the preceding paragraph. Few subjects have had a more "chequered career" than this one, and many mathematicians, from Leibniz onwards, have succumbed to the temptation to fall under its spell. Euler seems to have been the first mathematician to make any systematic use of divergent series, from which by formal manipulation he was able to produce many correct and important results. When, however, as was inevitable from time to time, he arrived at a conclusion, either partially incorrect or paradoxical, he frequently took refuge in metaphysics. A little later Poisson and Fourier continued their study: indeed Poisson's writings contain, in effect, the definition of Abel summability of a trigonometrical series. As the development of pure mathematics became more orthodox, however, along the lines followed by Cauchy and Abel, series which did not converge were gradually banished, and not until late in the nineteenth century did they reappear.

In two most valuable introductory chapters Hardy enlarges upon this theme, illustrating with numerous examples and quotations from the works of the great masters. He mentions, in particular, Abel as having written of divergent series as being "inventions of the devil", and explains how de Morgan, famous logician though he was, appears to have been quite incapable of appreciating the importance of definition, nowadays regarded as a commonplace. With his flair for seizing on the crucial point, Hardy sums up much of the confusion present in the minds of the early analysts as being due to their asking a question such as "What *is* $1-1+1-1+\ldots$?" instead of "How shall we *define* $1-1+1-1+\ldots$?" Also to be found in this introduction are accounts of Euler's work on the Riemann ζ-function (written in 1749, over 100 years before Riemann's own), Fourier's amazingly ingenious solution of an infinite set of linear equations in an infinity of unknowns, which led to the integral formulae for calculating the Fourier coefficients, and Heaviside's generalisations of the exponential and binomial series by means of Gamma functions.

Of the methods of summability that have been applied to divergent series, the simplest and in general the most important are those which are *regular*, in other words, those that sum all convergent series to their "ordinary"

sum. The two most familiar methods, usually denoted by $(C, 1)$ and (A), have this property and, moreover, in an extended sense, in that they also give an infinite sum for a properly divergent series. Irregular methods are, however, more numerous than one might expect, and one, in particular, used by Euler, leading to the paradoxical result $1+2+4+8+\ldots=-1$, is by no means insignificant, when applied with the necessary reserve.

In the third chapter the author discusses general theorems concerning regularity of summability methods involving linear transformations, originated by Toeplitz in 1911, and this is followed by another dealing with special methods of summation by means, including those of Nörlund, Euler, Abel and M. Riesz.

Two more chapters deal with the now familiar arithmetic-mean methods, that of Cesàro in which suitably weighted means are employed, and its equivalent, that of Hölder, where the weights are omitted. In the second of these chapters we first meet the idea of a Tauberian theorem, which recurs constantly throughout the rest of the book. It will be recalled that Tauber, in 1897, proved a *corrected* converse of Abel's theorem on power-series in which he showed that Abel summability together with the additional "Tauberian" condition $a_n=o(n^{-1})$ implies the convergence of Σa_n to its Abel sum. Later work by Littlewood in 1910 showed that the original Tauberian condition could be improved to read $a_n=O(n^{-1})$, an alternative and simpler proof of which was produced as recently as 1930 by Karamata. Chapter VII deals in detail with such theorems, and much of Hardy's own work is included therein. Chapters VIII and IX are concerned with the methods of Euler and Borel, and Chapter X with the multiplication of divergent series, including generalisations of the classical theorems of Cauchy and Mertens.

Another chapter on Hausdorff means leads us to one devoted to the Tauberian theorems of Wiener. These, of a character similar to but much deeper than their predecessors, are intimately connected with the theory of the Fourier transform, and it will be remembered that it was by their use that Wiener, Ikehara, Landau and others were able to provide elegant and much shortened proofs of the famous prime-number theorem.

In the last chapter, entitled "The Euler-Maclaurin Sum Formula", Hardy discusses the relations between asymptotic series and divergent series, with particular applications to the Riemann ζ-function and Stirling's theorem. A number of appendixes and lists of books, papers, periodicals and authors quoted serve to complete a really magnificent book, without which no library of pure mathematics could be regarded as complete.

It should be pointed out, in conclusion, that, as Hardy's death occurred before the last galleys had been sent to the press, the work on the page-proofs, bibliography and index fell on his associates, Professor W. W. Rogosinski, Dr. L. S. Bosanquet and Dr. H. G. Eggleston, and it redounds to their credit that this work was carried out in the spirit of and according to the high standards of their great master. J. H. PEARCE.

Several Complex Variables. By SALOMON BOCHNER and WILLIAM TED MARTIN. Pp. ix, 216. 40s. 1948. (Princeton University Press; Geoffrey Cumberlege, London)

The first encounter with Cauchy's theorem in the theory of functions of a single complex variable gives to most mathematicians a feeling of surprise that the values of a function on a contour should determine its values throughout the domain interior to the contour. Such inter-relationships become more intricate for functions of several complex variables: analyticity involves a whole set of differential equations on the real parts, or the imaginary parts, of such functions. Some of the resulting differences can be appreciated by

considering the repeated form of Cauchy's integral for functions of several variables. For one variable, the integral gives the values inside a domain in terms of the values on the boundary. In n variables, the values throughout a $2n$-dimensional domain in the space of the variables are given in terms of an n-fold line integral, hence in terms of an n-dimensioned subset of the boundary. There is not the one-one correlation between domain and determining set which we have in one variable, and it is easy to see that the line integrals which determine a function throughout a given domain may apply to a larger domain as well. Now in the ordinary complex plane, any region may be the complete domain of existence of an analytic function, with the frontier as a natural boundary. In n variables, it follows from the considerations given that to a given domain in the space of the variables there will correspond, in general, a larger domain, called its analytic completion, such that every function analytic in the first domain can be continued to be analytic throughout the analytic completion. These complications illustrate some of the difficulties of many variable theory, which is only at a very elementary level a repetition of one-variable theory.

Partly for such reasons, partly, perhaps, because it has not had the stimulus of problems such as those of elliptic and integral functions to give it direction, many variable theory has been slow to develop. Apart from preliminary work of Weierstrass and Poincaré, and, later, of Osgood and Hartogs, the main developments of the characteristic parts of the theory date back less than thirty years.

The book under review aims to give an account of some of the important parts of the theory of analytic functions of many variables: in spite of its title, it does not always confine itself to the case when all variables are complex, but frequently considers the effects of allowing some of them to take only real values. We may describe its approach as that which in one-variable theory is called Weierstrassian: the power series development is fundamental, while geometrical questions analogous to the Riemann theory, and, surprisingly in view of Bochner's own work, the Cauchy integral receive relatively little attention.

Chapter I deals with formal power series, ignoring questions of convergence. A set of n power series in n variables may be regarded as a formal mapping of the space on itself, and the sets for which all series have zero constant terms form a semi-group. Interesting theorems on groups and semi-groups of such transformations, the work of H. Cartan and others, are given. Chapter II begins the function theory proper, discussing some of the consequences which follow from the fact that a function is continuous and analytic in all its variables: the deep result of Hartogs, that analyticity in each variable implies continuity and analyticity as a function of all variables, is proved in Chapter VII. Chapter III continues the theory of Chapter I, but confines itself to mappings which are analytic functions. Chapters IV and V deal with the phenomenon of analytic completion, giving criteria for one domain to be the analytic completion of another.

In Chapter VI a number of inequalities analogous to those important in one-variable theory are proved, and Banach spaces of functions are discussed. This is preliminary to the development of Hartogs' theory in Chapter VII. By the use of these inequalities and by rather subtle set-theoretic arguments the theorems of Osgood and Hartogs which connect analyticity in each variable with analyticity in all are proved. A study of the radii of convergence of the developments of a function as a power series in one of its variables leads to the study of Hartogs' functions, which are much the same as those called plurisousharmoniques by Lelong and are a generalisation to several variables of subharmonic functions.

Chapter VII deals with topics in the theory of differential equations. Generalised solutions of a differential equation are defined to be functions satisfying a certain set of integral equations, which is such that every solution of the set which is differentiable sufficiently often is an ordinary solution. It is shown that for certain differential equations, of which Laplace's equation is typical, all generalised solutions are ordinary solutions.

Chapters IX and X rely on modern abstract algebra, the first in connection with the Weierstrass preparation theorem, and the second to discuss parametrization of irreducible analytic manifolds.

The authors say that " the book presupposes no knowledge of analytic functions of several variables, and there is no duplication of material presented in the books by S. Bergmann, by H. Behnke and P. Thullen, and by W. F. Osgood ", and that for the most part only a general knowledge of analysis and the elementary concepts of analytic functions of one variable are needed for its reading. The statements about the equipment required by the reader may be true for the larger part of the book ; but several sections require a knowledge of parts of modern abstract analysis, such as Haar's invariant measure on groups, Banach space theory (to a small extent) and Zorn's lemma. The style of the book is clear, but very terse, not to say austere. An amazing amount of material is to be found in its 212 pages of text. The austerity of presentation is perhaps accentuated by the lack of duplication mentioned, which means that the elementary parts of the subject get very small attention, and which probably has gone some way to determine the choice of subjects for this book.

There is no doubt that this book supplies an outstanding need : in English the only text-book on several complex variables is that of Forsyth (1914), and even in other languages there are no general accounts later than Osgood (1924) and the rather indigestible Ergebnisse memoir of Behnke and Thullen (1934). The book contains some material available nowhere else, and much only to be found in the original papers. It is all the more valuable because of the contributions of the authors, and particularly of Bochner, to the subject. The one criticism one may make is that the book might advantageously have been longer ; there is obviously a need for a comprehensive account of the subject, including the elementary material, in English. As it is, although the book is complete in itself a reader new to the subject might be well advised to consult some short elementary account, such as that in Valiron's *Cours*, if only for orientation before reading this work.

To conclude, this is a most important work, which will be indispensable to students of the subject. It maintains the high standards of the Princeton Mathematical Series, of which it is the tenth. The production, as usual for that series, is excellent. A slight criticism is that suffixes are not always clearly separated from symbols on the line : otherwise the printing is excellent, and appears notably free from error. J. L. B. COOPER.

Introduction to Complex Variables and Applications. By RUEL V. CHURCHILL. Pp. vi, 216. 30s. 1948. (McGraw-Hill)

This book, directed to engineers, physicists and applied mathematicians generally, is an excellent example of the skill in presenting mathematics to this audience which Professor Churchill has displayed in his other books. It covers the definitions of complex numbers and the complex plane, elementary functions, the analytic theory of functions and their geometrical theory considered as mappings of the complex plane, the Cauchy theory of complex line integrals and the derived theory of power series and residues, conformal mappings, and, more briefly, analytic continuation and Riemann surfaces.

Very little knowledge is assumed, and the essential definitions and proofs

are given clearly and carefully : for example, complex numbers, limits, and line integrals are all conscientiously defined.

In subject-matter the book differs from the normal text-book on complex variables mainly in the greater proportion of the space—almost one-third—devoted to conformal mappings. This, of course, corresponds to the importance of the subject in hydrodynamics and electrostatics.

We are familiar with books which seek simplification at the expense of rigour. Professor Churchill makes no such concessions. His proof of Cauchy's theorem, for example, is in Goursat's form and of a high standard of rigour. His presentation differs, however, from that of standard treatises in that he is at pains to illustrate general results by a large number of particular examples. The number of examples on conformal representation is particularly great, and the theory is well presented, though naturally without the general existence theorems. A good proportion of the examples in the text and the exercises are drawn from applied mathematics ; and a chapter is devoted to the Schwartz-Christoffel formula and its applications.

The last chapters, on Analytic Continuation and Riemann Surfaces, do not give proofs ; the ideas are illustrated by examples, and references are given to texts from which further details can be obtained.

The book is well produced and printed, and is very readable. It could be recommended, not only to the audience for which it is written, but to any mathematics student as an introductory text on complex variables. Certainly he would find its reading the best simple introduction to the elements of conformal mappings, as well as to other parts of the theory.

J. L. B. Cooper.

Cybernetics. By N. Wiener. Pp. 194. 18s. 1948. (Wiley, U.S.A. ; Chapman and Hall)

" If the seventeenth and early eighteenth centuries are the age of clocks, and the later eighteenth and the nineteenth centuries constitute the age of steam engines, the present time is the age of communication and control." This sentence quoted from the first chapter is the theme of Wiener's book. Its subtitle, " Control and communication in the animal and the machine," tells us that these ideas apply as much to the biological sciences as to engineering. The new subject of " information theory " arose at first from the communication engineering of the telegraph, telephone, and radio, but it has connections with many subjects. There is a connection with thermodynamics (or rather with statistical mechanics), because thermal fluctuations always interfere to some extent with any form of communication, and because where communication is concerned there is a difference between past and future which can be shown to have a thermodynamic origin. By regarding the newspapers, radio, and cinema as a form of communication there is a connection with sociology. Communication in the nervous system and by hormones bring in physiology and biochemistry. The transmission of hereditary characteristics by genes brings in genetics and evolution. There are also connections with the subject of " servo-mechanisms " and its physiological counterpart of " postural reflexes ". The word " Cybernetics ", coined by Wiener as a name for the whole subject, is derived from a word meaning " steersman ", and refers to the use of servo-mechanisms for automatic rudder control. Another important connection is with brains and automatic computing machines ; these are distinguished as systems in which the communication is almost entirely internal.

The main purpose of the book is to draw attention to these various connections, and to encourage workers in each of the fields involved to study the others. It is written with an admirable boldness of style. Wiener has rightly

decided that in such a book it is better to risk criticism on points of detail than to obscure the meaning with many qualifications, or to avoid unproved conjectures. There is little space for the detailed consideration of all the connections. Since Wiener is himself a mathematician, it is natural that the connections with thermodynamics are considered in greatest detail. Unfortunately the mathematical parts of the book have many misprints: sufficiently many to be a serious hindrance to understanding at many points. A few references are given, but not sufficient to enable the many stimulating suggestions to be followed up easily. A. M. T.

Les Principes de l'Analyse Géométrique. Par GEORGES BOULIGAND. Tome I, Leçons de Géométrie Vectorielle : Préliminaires à l'étude de la théorie d'Einstein. 3ième ed. Pp. x, 436. 1200 fr. 1949. (Librairie Vuibert, Paris)

In his stimulating preface to the third and revised edition of Prof. Bouligand's book, Prof. Goursat explains that the French have been slow in adopting vector methods because there is no standard notation. One of the merits of this volume is that the cumbersome continental symbol with an arrow on top has been rejected in favour of the Anglo-Saxon symbol in Clarendon type. Moreover, the dot and cross ($\wedge$) symbols have been used to denote scalar and vector products respectively. The one remaining point of difference between the notation now generally accepted in this country and that used in this book is the retention of *rot* for our *curl*. It is to be hoped that at the forthcoming International Congress of Mathematicians an attempt will be made to standardise vector notation definitively and universally.

The present volume is divided into three parts : vector operations in linear (affine) geometry ; vector operations in metric geometry ; vector calculus. There are five appendices, running to about 130 pages, on the principles of tensor calculus, Riemannian geometry, the foundations of geometry, surfaces of constant curvature, variants and extensions of vector methods (including quaternions and Grassmann's calculus), respectively. The treatment, which begins from first principles, is thorough and, in many respects, novel. For example, the ideas of angle and volume are introduced in a refreshingly original way. The relation between the angle θ at O, as defined by the author, and the corresponding arc length of the unit circle with centre O is shown to depend on the solution of the functional equation,

$$f(\theta)+f(\theta')=f(\theta+\theta').$$

The discussion of the concept of volume $V(P)$ of a polyhedron is particularly interesting. The author defines this so that (i) if P is decomposed into two non-overlapping polyhedra P_1 and P_2, $V(P)=V(P_1)+V(P_2)$, (ii) $V(P)$ is unaltered by a displacement, *e.g.* a translation, of P, and (iii) the volume V of a parallelepiped based on three position vectors **OA, OB, OC** is a continuous function of these vectors. These definitions are shown to characterise V as a "trilinear alternate function" of **OA, OB, OC,** having the same properties as the standard determinant formed by the nine components of these three vectors. Incidentally, the author shows how this technique yields an elegant proof, similar to that in Courant's well-known book, of the rule for the multiplication of determinants of order n.

The outstanding feature of Prof. Bouligand's treatise is its logical clarity. For example, he concludes his account of linear (affine) geometry with a discussion of its logical autonomy. He shows that it is based on two concepts, points and free vectors, each being defined by definite properties. In particular, free vectors are defined so that the sum of any two yields a unique third vector, the product of any vector **a** by a scalar λ a unique vector λ**a**, the usual laws of commutation, association and distribution being obeyed.

Finally, if the dimension of the system is n, any free vector in it can be uniquely expressed as $\sum_{r=1}^{n} \lambda_r \mathbf{a}_r$, where $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ is a fundamental set of vectors.

Prof. Bouligand's treatment of vectorial geometry contrasts sharply with Prof. Milne's use of vector technique in his recent book, *Vectorial Mechanics*. The theme of the latter is the vector product $\mathbf{a} \wedge \mathbf{b}$ and its applications, whilst in the former this powerful concept plays only a subsidiary role. In particular, continued, or triple, vector products are never mentioned.

Although Prof. Bouligand has devised an interesting new line of approach to tensor calculus, the reviewer is not convinced that it is an essential improvement over other methods. In brief, having shown how every linear vector function can be represented by a " doublet ", *i.e.* by an ordered pair of parallel planes, the author defines a tensor as a function of any number of elements, vectors or doublets, depending linearly on each of them. He discusses in considerable detail the principles of vector calculus and its application to the classical differential geometry of curves and surfaces, in particular " *la méthode du trièdre mobile* ". The appendices contain a useful introduction to Riemannian geometry for the benefit of the prospective student of relativity. This introduction meets with the full approval of the author's distinguished sponsor, Prof. Goursat, who points out that :

" Quoique M. Bouligand ne parle à aucun moment, à ce qu'il me semble, de la relativité, je crois pouvoir affirmer qu'en rédigeant les notes qui terminent le volume, et en particulier les deux premières, il a pensé aux étudiants desireux de s'initier à ces nouvelles théories. . . .

" J'ai entendu quelquefois, avec quelque étonnement, des personnes sans aucune culture scientifique, déclarant de bonne foi (du moins, je veux le croire) qu'elles avaient compris la relativité. M. Bouligand, et je l'en félicite, n'est pas de ces auteurs qui ont la prétention d'introduire le premier venu, sans initiation mathématique préalable, dans ces profondes théories. Il sait fort bien que tout mathématicien qui veut écrire sur un sujet nouveau et difficile a deux devoirs essentiels à remplir. Tout d'abord, il ne doit pas ajouter aux difficultés inhérentes au sujet lui-même des difficultés accessoires. . . . Le second devoir . . . est de bien mettre en évidence les veritables difficultés de la question, et non de chercher a les dissimuler par des artifices plus ou moins brillants. Le rôle de l'auteur est d'éclairer le lecteur, non de l'éblouir par sa virtuosité. L'exposition de M. Bouligand me paraît satisfaire à ces deux conditions. . . ."

The author announces his intention to publish a sequel to the present volume for the benefit of pure mathematicians rather than of mathematical physicists. He will concentrate on the theory of sets, the theory of groups, and topological concepts in geometry. On this prospective volume he comments picturesquely : " un changement de climat se produira donc, analogue à celui qu'un chimiste observe en passant des réactions intermoléculaires à celles du noyau! "

G. J. W.

Géométrie Descriptive. Par GASPARD MONGE. Translated into Russian by W. F. HAASE. Pp. 291. Rs. 20. 1947. (The Press of the Academy of Sciences of the U.S.S.R.)

This, probably the first, translation of this fundamental work into Russian was prepared from its third French edition of 1811. The Russian version contains the original diagrams, besides reproductions of two portraits of Monge and of the title page of his book bearing the date " An VII "—the seventh year of the Republic.

The text is followed by a commentary, a short biography and an apprecia-

tion of the contribution made by Monge to the study of Geometry. These were prepared by the editor, Professor D. I. Kargin.

The most valuable part of the book, in so far as the Western reader is concerned, is a list of 72 French titles of publications bearing Monge's name (pp. 258–264) and a bibliography of 73 French, English, Russian and German references to his biography and work (pp. 264–270). These lists were prepared by A. M. Lukomskaya of the library of the Academy.

The book under review is an addition to a collection which is being published under the general title "Classics of Learning". The appearance of this particular volume is timed to coincide with the 150th anniversary of the publication of the original. L. M. T.

The Modern Approach to Descartes' Problem. By SIR EDMUND WHITTAKER. Pp. 30. 1s. 6d. 1948. (Nelson)

This booklet contains the Herbert Spencer Lecture delivered by Professor Whittaker in the University of Oxford in 1948, and deals with the relation of the mathematical and physical sciences to philosophy.

Since the days of Aristotle it has been a cherished ambition of mankind to construct a comprehensive system of philosophy competent to explain in terms of primitive logical elements the whole of life and to embrace, not only the inorganic and the material, but also the organic and the spiritual. Impressed as much by the sterility of contemporary philosophical disputation as by the solid achievements of mathematics in the seventeenth century, Descartes was led to the belief that the intuitive procedure of the past should be replaced in some way by a formulation based on the stricter methods of mathematics. How to do so was the "Problem of Descartes". Though his attempt to solve it has played a significant part in the creation of modern science, such matters as human consciousness, purpose and sense of value eluded his scheme and he failed in his main objective.

A situation broadly similar to that which confronted Descartes exists at the present time. While spectacular advances have been made in the mathematical sciences, philosophy not only continues to lack unanimity but seems to have lost the initiative in intellectual progress. Indeed, the impacts of these sciences upon modern philosophy have been as severe as they have been numerous. The concept of a space-time continuum, the doctrine of transfinite numbers, the dual behaviour of the electron, the vagaries of radiative phenomena, the creation of a non-Aristotelian three-valued logic together with Peano's symbolic calculus, and much besides, have all exercised a disturbing, if stimulating, influence on philosophic thought.

Professor Whittaker suggests that the time is ripe for a fresh attack, in the light of this new knowledge, upon the Problem of Descartes, and that "the promised land which Aristotle and Descartes saw from afar is now open to be entered". He sketches in brief outline a doctrine of space and introduces a new concept of reality which includes relationships essentially mathematical in character. The numerical ratios associated with the musical scale provide a link between reality, aesthetic value and the human consciousness. The ground is thus prepared for the creation of a *Neo-Cartesianism* which will recognise moral values and acknowledge the idea of God.

Clearly this is a thought-provoking little book and characterised by that breadth of vision, felicity of style and clarity of presentation, for which its author is so justly celebrated. T. A. B.

An Introduction to the Laplace Transform. By J. C. JAEGER. Pp. vii, 130. 7s. 6d. 1949. (Methuen's Monographs on Physical Subjects)

The need for brevity imposed by these monographs has especially happy

consequences for a subject where it is easy to lose sight of the wood of a straightforward method of attack on differential equations for the arboreal array of manipulative theorems and considerations of the transform for its own sake. In this small book the use, and to a lesser extent the theory of the Laplace transform are carried " as far as possible using no mathematics beyond the ordinary calculus, but at the same time catering for the reader who wishes to acquire the manipulative skill to solve his own problems ". At the same time, questions such as convergence and the use of the theory of functions of a complex variable are mentioned often enough to leave no doubt of their importance.

The preliminary properties considered before the main body of applications of the transform (largely to electrical problems) are confined to twenty-odd pages, and the balance restored in a later chapter. This deals with and applies shift and superposition theorems, and the Mellin inversion theorem is stated. The final chapter is devoted to partial differential equations, exemplified by those concerned with electrical transmission lines.

The book appeared in a mimeographed edition in Australia in 1945, and makes a good readable introduction to the method ; a number of examples are provided for practice in each chapter. R. B. H.

Invariante Kennzeichnung relativ-abelscher Zahlkörper mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. By H. HASSE. Pp. 56. DM. 6. 1949. (Akademie-Verlag, Berlin)

This memoir is concerned with the very general and deep question of the extension of a field Ω to a field K, the relative group being a given abelian group A, and with the invariant characterisation of such an extension.

H. D.

The Annals of the Computation Laboratory of Harvard University. III.

1. Vol. IX. **Tables of the Bessel Functions of the First Kind of Orders Sixteen through Twenty-seven.** Pp. xii + 764. 1948.

2. Vol. X. **Tables of the Bessel Functions of the First Kind of Orders Twenty-eight through Thirty-nine.** Pp. xii + 694. 1948.

3. Vol. XI. **Tables of the Bessel Functions of the First Kind of Orders Forty through Fifty-one.** Pp. xii + 620. 1948.

4. Vol. XVI. **Proceedings of a Symposium on Large-scale Digital Calculating Machinery.** Pp. xxx + 302 + 1 plate. 1948.

5. Vol. XVII. **Tables for the Design of Missiles.** Pp. lvi + 226. 1948.

Vols. IX, X, XI and XVII are by the STAFF OF THE COMPUTATION LABORATORY, Cambridge, Mass. : all published by the Harvard University Press ; London : Geoffrey Cumberlege, Oxford University Press. 10·4″ × 7·8″ (26·4 cm. × 19·8 cm.). 63s. each volume.

Earlier volumes in this series were reviewed in *Math. Gazette*, **30**, 1947, pp. 178–181 ; **33**, 1949, p. 73.

Volumes IX to XI continue the series of *Tables of Bessel Functions of the First Kind* which started with Vol. III and which will presumably extend to Vol. XV (Vols. XII to XV have not yet appeared). Many of the remarks made in the earlier reviews apply to these volumes also, though they are a little more convenient in that they contain more orders per volume.

Although the reviewer strongly favours printing from type for most tables, there is a field, mainly concerned with tables of restricted application, where photographic methods are more appropriate. The present volumes IX, X, XI and XVII fall within this field, and the reproduction is unusually and uniformly good.

(1) (2) (3). Volume IX tabulates $J_n(x)$ to 10 dec. for $n = 16(1)27$ and $x = x_n(0{\cdot}01)99{\cdot}9$ where x_n is the last argument for which $J_n(x) < \frac{1}{2} \times 10^{-10}$. Volumes X and XI do the same for $n = 28(1)51$, except that the early values for each order are given as critical tables.

The form of critical table is such that it needs great care to interpret it correctly. Thus for J_{28}, J_{29}, one has the table alongside (here given in units of the 10th decimal, though all zeros, including one before the decimal point, are given throughout the tables reviewed). It is presumably implied that $x = 10{\cdot}43$ is the *first* x for which $J_{28} = 2 \times 10^{-10}$; this is not stated, nor is it clear from the table. The "interference" between columns is curious, though sometimes helpful. For example, it is due to this that 11·10 is seen certainly to be the *first* argument for which $J_{29} = 2 \times 10^{-10}$, $x = 11{\cdot}09$ would not have occurred apart from its need in the table of J_{28}.

x	$J_{28}(x)$	$J_{29}(x)$
9·99	0	
10·00	1	
10·43	2	
10·63	3	
10·64	3	0
10·65	3	1
10·77	4	1
10·88	5	1
10·96	6	1
11·03	7	1
11·09	8	1
11·10	8	2

No introductory text, apart from a brief Preface, occurs in these volumes, the reader being referred to Vols. III and V.

Vol. XI is printed on slightly thinner paper than earlier volumes, and since it has also fewer pages than Volumes III to X, it is rather more convenient to handle.

(4) Some quotations from Preface of this Volume (by Prof. H. H. Aiken, Director of the Harvard Computation Laboratory) will explain its purpose.

"The Staff of the Computation Laboratory has long felt the need of better channels of communication among those mathematicians, physicists and engineers interested in the design, construction, operation and application of large-scale digital calculating machinery. Hence, late in 1946, it seemed certain that a meeting of this group would materially advance the art of computation. . . .

"Plans were therefore made by the Bureau of Ordnance and Harvard University to sponsor jointly a symposium on large-scale digital calculating machinery to be held in the new Computation Laboratory of Harvard University from the seventh through the tenth of January, 1947.

"The topics for the several meetings were selected with a view to discussing those portions of the field most pertinent at the time. . . .

"During the symposium a great many people expressed a desire to obtain copies of the speeches. This general and widespread interest led the Staff to believe that the proceedings of the Symposium should be published as a volume of the Annals of the Computation Laboratory."

The eight sessions of the symposium were each devoted to a single broad topic, except the first and last, which consisted respectively of *Opening Addresses* and *Conclusions and Open Discussion*.

The second session dealt with *Existing Calculating Machines*, with interesting papers on *The Work of Charles Babbage*, by Richard H. Babbage, great-grandson of Charles Babbage; on the Harvard *Mark I Calculator* by Richard M. Bloch, on a *Brief Description and Operating Characteristics of the ENIAC* by Lewis P. Tabor, on the *Bell Telephone Laboratories' Relay Computing System* by Samuel P. Williams, and on the Harvard *Mark II Calculator* by Robert V. D. Campbell.

Other sessions dealt with *The Logic of Large-Scale Calculating Machinery, Storage Devices, Numerical Methods and Suggested Problems for Solution,*

Sequencing, Coding, and Problem Preparation, and *Input and Output Devices*.

There are many papers of interest, but only two, which deal with a topic of particular interest to the reviewer, will be mentioned; these are *Problems of Mathematical Analysis involved in Machine Computations* by Dr. Alexander W. Wundheiler and *On the Accumulation of Errors in Processes of Integration on High-Speed Calculating Machines* by Prof. Hans A. Rademacher. Both are concerned with the accumulation of rounding-off errors during a succession of arithmetical operations; the former paper, while it raises several points of interest and importance, contains statements and suggestions that will hardly convince an experienced computer!

On the other hand, Prof. Rademacher's important paper is a scholarly attack on the problem of estimating statistically the accumulation of errors in two particular specimen cases. Both truncation errors, due to use of an approximate integration formula, and rounding-off errors are considered. To the reviewer, the only proper basis for discussion of computation errors appears to be the *maximum* error; nevertheless, it may be that in electronic computation, the number of successive repetitions of processes makes a statistical approach more desirable, since use of maximum error would mean carrying perhaps two extra figures in the computation, in addition to the guard figures retained.

These two papers may perhaps give an indication of the rather wide range of merit of the contributions to the symposium. Nevertheless, the volume is an interesting one, and should be examined by all who like to keep up with modern computing developments.

(5) As indicated by its title, this is a table for specialists. It gives tables of auxiliary functions needed in calculations of volumes, positions of mass centres and moments of inertia of projectiles in the form of solids of revolution obtained by rotation of a plane figure, with boundary composed of straight lines or arcs of circles, about an axis in its plane.

It will suffice to indicate a few of the functions tabulated:

$$H_1=\frac{20}{a^5}\int_0^a (1-\sqrt{1-x^2})^2dx=\frac{20}{a^5}\left[2a-\sin^{-1}a-a\sqrt{1-a^2}-\frac{1}{3}a^3\right],$$

$$H_2=\frac{24}{a^6}\int_0^a x(1-\sqrt{1-x^2})^2dx=\frac{24}{a^6}\left[a^2-\frac{1}{4}a^4-\frac{2}{3}+\frac{2}{3}(1-a^2)^{\frac{3}{2}}\right],$$

$$H_3=\frac{28}{a^7}\int_0^a x^2(1-\sqrt{1-x^2})^2dx$$

$$=\frac{28}{a^7}\left[\frac{2}{3}a^3-\frac{1}{4}\sin^{-1}a+\frac{1}{4}a\sqrt{1-a^2}-\frac{1}{2}a^3\sqrt{1-a^2}-\frac{1}{5}a^5\right],$$

$$H_4=\frac{144}{a^9}\int_0^a (1-\sqrt{1-x^2})^4dx$$

$$=\frac{144}{a^9}\left[8a-\frac{8}{3}a^3+\frac{1}{5}a^5-\frac{7}{2}\sin^{-1}a-\frac{9}{2}a\sqrt{1-a^2}+a^3\sqrt{1-a^2}\right].$$

These are tabulated to 6 decimals, with first differences, for

$$a=0(0{\cdot}001)0{\cdot}999.$$

Certain functions, of two variables, depending on the H_r are also extensively tabulated; for example,

$$s=1-\frac{a^2}{3(1-m)}+\frac{ma^4}{20(1-m^2)}H_1,$$

and others more complicated. There are also tables to 6 decimals, and for $x = 0(0{\cdot}01)0{\cdot}99$, of

$$s = \frac{1}{3}(x^2 + x + 1), \quad t = \frac{1}{6s}(3x^2 + 2x + 1),$$

$$u = \frac{1}{5s}(x^4 + x^3 + x^2 + x + 1), \quad v = \frac{1}{20s^2}[(1 + x)^4 + 4x^4].$$

An extensive introduction gives formulae and deals with the construction and use of the tables. J. C. P. MILLER.

Mathematics of Statistics. II. By J. F. Kenney. Pp. ix, 202. 15s. 1939; rep. 1947. (Van Nostrand, New York; Macmillan, London)

The second edition of Part I of this book, published 1947, was reviewed in the *Gazette* for February 1949, XXXIII, No. 303, p. 68. We have now the seventh printing of Part II.

It is apparently as first published, but has not previously been reviewed in these pages. The purposes of the two books are given by the author as follows: "There are two quite distinct aspects or levels of mathematical statistics. The one involves elementary mathematics and the methodologies serve descriptive purposes. These fundamentals are set forth in Part I. The other aspect is essentially mathematical in character and the methodologies are developed for inferential purposes. . . . In this book (Part II) I have elected to include some of the classical theory and some of the simpler concepts and techniques of the modern theory." In the Preface to Part I he further explains that Part II contains a systematic treatment of the theory of sampling. The present book is therefore largely independent of Part I: the chapters are numbered from a fresh start, and there are very few cross-references.

The book is rather confusing. Thus the distribution function $f(x)$ is defined in Chapter III, on p. 43, but has already been used in Chapter I. It is not clear how the universe of p. 107 gives what is called the type III curve of this universe on p. 108, with its abrupt lower end. There is no legend to the graphs on p. 47. "Typical Curves of the Pearson System," to explain the various ways in which Pearson described or derived them. *Null hypothesis* is twice defined (once on p. 117, and again on p. 181); *Statistic* similarly is defined twice (on p. 98 and on p. 125). The treatment is not too rigorous: it is not explained what is meant by (p. 50) the observed measurements satisfying approximately the relation given. On p. 171, not interpolation but extrapolation is surely required to deal with chi-squared as 5·8 for $n = 7$. Some of the examples are rather unreal: thus (p. 89) "suppose . . . solution $r_{1\cdot23} = 1{\cdot}29$," and (p. 90) "find . . . in order that $r_{1\cdot23} = 1$."

As in Part I, references are badly made. Yule and Kendall is in the Index, referring us to p. 95. There we find an example, "Yule and Kendall" with no explanation about the two names, their book, the page—nothing else at all. Other references, such as those in the bibliographies at the end of several of the chapters are given without date, a serious disadvantage in a book reprinted in 1947 without any revision (apparently) since its first appearance in 1939, particularly in referring to a book such as Fisher's *Statistical Methods for Research Workers*, which has gone through so many editions, each with considerable modification on its predecessor. References such as (pp. 153, 162) to "Mills' text, revised" are not of much value in the absence of any explanation, as far as we can trace, of what this is. *Tables for Statisticians and Biometricians* is an insufficient reference to a work that was published in two parts, the first in 1914 and the second in 1931.

All this is a pity, for there is much that is good in the book. It would serve well as a note book. It gets near to being quite good in its treatment

of the derivation of the chi-squared test, of Tchebycheff's inequality, of multiple and partial correlation and of fiducial limits. The three tables given (of normal curve, of Snedecor's F, and of chi-squared) are all useful. But the whole book should have been rewritten and not merely reprinted.

FRANK SANDON.

School Geometry and Trigonometry. By C. V. DURELL. Part I. Pp. 176. 3s. 9d. Part II. Pp. 144. 3s. 6d. Part III. Pp. 144. 3s. 6d. Parts I and II together, 6s. 6d. Complete in one volume, 8s. 6d.; with answers, 9s. 1949. (Bell)

The volume provides a unified course of (Stage B) geometry and trigonometry, divided into three Parts, each of which corresponds to a year's work. Part I covers the usual ground in geometry as far as Pythagoras' Theorem, with similarity brought into prominence, and trigonometry taken as far as the cosine formula. Part II covers loci, the circle, the solution of triangles by the sine and cosine formulae, and three-dimensional trigonometry. Part III covers ratio and similar figures, the bisector theorem, mensuration, the drawing of Plans and Elevations, and Navigation. The level reached is that of Additional Mathematics in School Certificate, but some of the exercises touch on more advanced topics, such as Ptolemy's theorem, Apollonius' circle and the focus-directrix loci; and in addition some of the work in Solid Geometry and Navigation is perhaps a little beyond that level.

The author draws attention in his preface to the following features of the book: (1) the use of trigonometry as an invaluable geometrical weapon, (2) the restriction of formal proofs of theorems to a few key theorems, (3) the close association between tests for congruence and tests for similarity, (4) applications to solid geometry throughout the course, (5) navigation, (6) summaries at the end of each Part.

Other excellent features impress one in reading the book. Each new theorem or property is introduced with a thorough class discussion which brings out both the property and its proof by means of a skeleton argument to be filled in. Then follow numerical examples and exercises, which gradually become more general, and the formal proof (when given as such) comes as a culmination to be followed only by riders, in which hints are often given suggesting a helpful construction. This arrangement and grading of the exercises, together with the Revision Papers at the end of each Part, are admirable.

The trigonometry is well integrated with the pure geometry. For example, in the exhaustive class discussion leading to Pythagoras' Theorem, the cosine formula is obtained first and the main theorem deduced therefrom, though later Euclid's proof and that by similarity are given. The book shows well the advantage in comprehensiveness of thus combining geometry and trigonometry. Formal trigonometrical notation is not introduced until a thorough understanding of the ratio properties of similar triangles has been obtained. For this the criteria for similarity of triangles are taken as equally fundamental with those of congruence. The discussion on these criteria could well have been expanded, but the author has assumed this matter to have been fully considered in the Stage A course. There will be differences of opinion about the logical soundness and practical wisdom of thus raising the status of similarity to the same level as that of congruence—differences that were expressed in the discussion on the Teaching of Geometry at the Annual General Meeting this year. An indication of this change from the traditional order (and from the author's attitude in earlier works) is that the mid-point and intercept theorems are proved by similarity and not congruence. There is no attempt at economy of postulate, and no suggestion anywhere in the

book that the criteria for similarity may be derived from those for congruence. This would perhaps be hardly necessary (or to be desired nowadays) in a book designed for average pupils, but the omission is surprising in this book otherwise so comprehensive and suitable only for able pupils.

Only eleven properties are dignified by the name of Theorems; all of the others are called Useful Facts, although they are fully proved. This distinction between "key" theorems and others helps a pupil to see the development of the subject, and everyone would agree with most of the eleven chosen, though perhaps not with all. The precise choice, however, is sometimes a little perplexing. For example, the Summary to Part I (pp. 173–6) gives the angle in a semicircle as a Useful Fact, but not the relation between the angles at the centre and the circumference, although it has been proved on pp. 85–6 as an example on isosceles triangles, and is used later on p. 179 before it has received formal treatment on p. 219. The reason is, no doubt, that later, in the Summary to Part II, it is elevated to the status of Theorem. With a few other topics besides this, the thoroughness of method leads sometimes to repetition, but this is not a serious fault in a text-book.

Two of the chapters in Part III, viz. Chap. 15 on Ratios and Chap. 18 on Mensuration, could with advantage have been placed much earlier in the book. The former of these, taken early in Part I, would have provided a still sounder basis for trigonometry, and the latter chapter, placed early in Part II, would have enabled some of the harder work of Part II to be postponed a little.

There are a few small mistakes of order: *e.g.* the word "transversal" is defined on p. 329 but had already been used on p. 8, and the "distance of a point from a line" is defined on p. 262 but had been needed in riders on pp. 182 and 257.

A few of the diagrams are slightly inaccurate or misleading (though perhaps more so to a teacher than to a pupil), viz. Figs. 100, 101, 116, 270, 452, 495; and there are several type blemishes, but most of the typography is good.

The part of the book on Solid Geometry includes, besides the usual work, oblique and isometric projection and a full chapter on Plan and Elevation. The chapter on Navigation covers Magnetic Variation and Deviation, the triangle of velocities and related problems. These two chapters stand rather apart from the closely-knit texture of the rest of the book, but deal with their subjects very competently.

There are tables of logarithms, logarithmic and natural sines, cosines and tangents, squares and square roots, and an index.

The criticisms made here have been mainly on the logical background of the book, and do not detract appreciably from its practical value to an able pupil. It is not suitable, however, for less able children. For them it is too thorough and the appearance of the pages perhaps not attractive enough. However, the success of the book as a unified course stimulates one to consider what would be the most perfect arrangement for teaching purposes. W. O. S.

Mathematics. By T. H. Ward Hill. Pp. 343. 10s. 6d. **1949.** Torch Books, 2. (Harrap)

"The elements of mathematics are here simply explained for the benefit of a non-technical public and the book will serve either as an introduction to the subject or as a 'refresher' course." So runs the statement on the cover. It is a remarkable claim in a book of large print and 300 comparatively small pages. There is no royal road to mathematics, and only in a very broad sense is it possible to regard this book as an "introduction to the subject".

The author is interested in mathematical history. All six parts contain allusions. Part I introduces us to arithmetic through the ages, passing on to directed numbers, fractions and decimals. We hear of L.C.M.'s and are told

"the old-fashioned way is probably still the best" way of finding it! We learn how to multiply and divide, with a page from Robert Recorde's book to convince us, presumably, how much easier things are to-day, and this section is rounded off with "Applied Arithmetic", which means little more than a few pages devoted to ratio, interest and stocks.

With his insistence on the importance of history, it is a misfortune that the author is not equally insistent on accuracy. In the geometrical section we are told that Shanks gave the value of π correct to 707 places in 1853. Actually he did not publish the 707 places till 1873, although 607 places were given in 1853, and his result was only correct to about 527 places, when he made a mistake, as has been recently shown by Mr. D. F. Ferguson.

After forty pages of algebra we are introduced to logarithms and calculating machines, though here the poor bewildered student will have some difficulty! "Peeps over the Fence," the title of Part VI, sounds romantic, and in twenty pages we learn vector addition, the addition formula in trigonometry, statistics and the calculus!

Finally, we have the conclusion to comfort us, if we have been preparing for the School Certificate. "In Arithmetic the course has been almost covered, certainly the principles; in algebra and trigonometry much of it has been covered; in geometry...." This is perhaps the most striking example of wishful thinking we have encountered. To learn mathematics we need more aids than are given in this book. W. F. BUSHELL.

Number Stories of Long Ago. By DAVID EUGENE SMITH. New ed. Pp. viii, 158. 3s. 6d. (Ginn)

This is the first English edition of the well-known book published in America in 1919. There are some slight differences. The cover is more attractive. The print and pages are a little larger, and, although the coloured pictures are omitted, there are a few extra diagrams in the text to take their place.

The author quotes Plato: "Do not then train boys to learning by force and harshness; but direct them to it by what amuses their minds." So he provides a story-teller, who is represented as talking to children about the gradual invention of numbers, in accordance with man's needs, up to the Hindu-Arabic numerals of to-day.

We read of the Egyptians and Assyrians and, of course, of the Greeks and Romans. Addition and Subtraction followed fairly readily, but multiplication and division were a stumbling-block for many centuries until the Arabic numerals came into common use. We tend to forget that even Robert Recorde some 400 years ago "used his influence to make people give up the Roman numerals". Fractions and even decimals are mentioned, as is right, because the former occur in the Rhind papyrus, and indeed the simplest were in common use for millenia; but decimals can hardly be said to be anything but modern. Indeed the author reminds us that the first book published on decimal fractions is only a little more than three centuries old.

We are glad that he pays a due tribute to Cuthbert Tonstall, who in 1522 published the first Arithmetic printed in England. His rules for multiplication are very much like ours, though he works rather ponderously with the zero.

The latter third of the book deals with some famous number problems, starting with magic squares. Most of them have a long ancestry. In a note to teachers the author claims to have selected them from the considerable literature on the subject dating back to Greek mathematics. He reminds us that the work is not written for scholars, and the historical facts are therefore set out as well as such circumstances allow. A more scholarly book would inevitably have been of much greater length.

But the author shows skill and imagination in his choice. His name is sufficient guarantee of accuracy, and although the form in which it is put, and the occasional badinage between the story-teller and the children may be distasteful to some, yet it is certain that all teachers will here find something to help them in their work and stimulate them to further research.

W. F. BUSHELL.

Integralgleichungen. Einführung in Lehre und Gebrauch. By G. HAMEL. 2nd edition. Pp. vii, 166. DM. 15.60. 1949. (Springer, Berlin)

This new edition is a corrected reprint of the first (1937) edition, but although it contains no new material a new issue is very welcome. The book is a most pleasant and readable introduction to the elements of the theory of integral equations, and should be of particular value to the increasing number of students who need a sound knowledge of fairly advanced mathematics for their attack on problems of mathematical physics and technical applications. English textbook literature on the topic is scanty, and of the numerous German accounts, Hamel's is perhaps the easiest to read.

The theory is brought in gradually by means of examples from vibration problems, potential theory, and so on, without any premature straining after generality. The methods of Fredholm, Schmidt and Hilbert are sketched, and in a final section the author returns again to special problems, some of interest chiefly theoretical, some arising from physical investigations. The excellent survey of fundamental theorems, and the wealth of interesting applications, set forth in a pleasant style not too concise for the novice, make this a most useful little volume. T. A. A. B.

CORRESPONDENCE.

TCHEBYCHEFF'S FORMULA FOR NUMERICAL INTEGRATION

To the Editor of the *Mathematical Gazette*.

SIR,—On p. 50 of the *Gazette* for February 1949, Professor A. C. Aitken reviews Fort's book on Finite Differences, and mentions that Fort refers to Bernstein's proof that some of the n abscissae used in Tchebycheff's formula may be imaginary when $n > 9$.

In a paper on the motion and strength of ships, published in the *Transactions of the Institution of Naval Architects* for 1896, Captain A. Kriloff uses Tchebycheff's formula and states that even when $n = 8$ some of the abscissae are imaginary. In an earlier paper he had contributed in 1893 to the *Bulletin de l'Association Technique et Maritime*, he makes the same remark, and shows how the abscissae may be found fairly easily from Newton's formulae for the sums of powers of roots. He refers therein to Tchebycheff's original paper, "Sur les quadratures," (*Journal de Liouville*, 1874) but this does not consider numerical cases when n is greater than 7.

It seems that Kriloff's work, published in technical journals, may have escaped the attention of mathematicians.

I may add that many naval architects use 8 or 10 ordinates, and to avoid imaginaries, take $n = 4$ or 5 twice over.

Yours, etc., L. WOOLLARD.

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ILLUSTRISSIMÆ

SOCIETATI REGALI

a Sereniſſimo

REGE CAROLO II.

AD

PHILOSOPHIAM PROMOVENDAM

FUNDATÆ.

ET AUSPICIIS

POTENTISSIMI MONARCHÆ

JACOBI II.

FLORENTI.

Tractatum hunc humillime *D. D. D.*

JS. NEWTON.

C.56.h.4